Laplace spectra on open and compact Zeeman manifolds

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Abstract

By a recent observation, the Laplacians on the Riemannian manifolds the author used for isospectrality constructions are nothing but the Zeeman-Hamilton operators of free charged particles. These manifolds can be considered as prototypes of the so called Zeeman manifolds. This observation allows to develop a spectral theory both on open Z-manifolds and their compact submanifolds.

The theory on open manifolds leads to a new nonperturbative approach to the infinities of QED. This idea exploits that the quantum Hilbert space, $\mathcal{H}$, decomposes into subspaces (Zeeman zones) which are invariant under the actions both of this Zeeman-Laplace operator and the natural Heisenberg group representation. Thus a well defined particle theory and zonal geometry can be developed on each zone separately. The most surprising result is that quantities divergent on the global setting are finite on the zonal setting. Even the zonal Feynman integral is well defined. The results include explicit computations of objects such as the zonal spectra, the waves defining the zonal point-spreads, the zonal Wiener-Kac resp. Dirac-Feynman flows, and the corresponding partition functions.

The observation adds new view-point also to the problem of finding intertwining operators by which isospectral pairs of metrics with different local geometries on compact submanifolds can be constructed. Among the examples the author constructed the most surprising are the isospectrality families containing both homogeneous and locally inhomogeneous metrics. The observation provides even quantum physical interpretation to the isospectrality.

1 Zeeman manifolds

Zeeman-Hamilton operators. The classical Zeeman operator of a charged particle is

$$H_Z = -\frac{\hbar^2}{2\mu}\Delta_{(x,y)} - \frac{\hbar eB}{2\mu c\mathbf{i}}D_z \bullet + \frac{e^2B^2}{8\mu c^2}(x^2+y^2) + eV, \tag{1}$$

where V is the Coulomb potential originated from the nucleus (for free particles $V=0$ holds). At the time it was introduced, the new feature of this operator was the orbital angular momentum operator $D_z\bullet = x\partial_y - y\partial_x$, which was the forerunner for an adequate spin-concept. This operator is the result of a long agonizing creative process [To], which was used for explaining the Zeeman effect. (Note that the $D_z\bullet$ commutes with the rest part, $\mathbf{O}$, of the complete operator. Thus the spectrum appears on common eigenfunctions, resulting that the $D_z\bullet$ splits the spectral lines of $\mathbf{O}$ (Zeeman effect).) Pauli, who added a spin angular momentum operator to the orbital one, developed the non-relativistic spin-concept. The relativistic concept due to Dirac. Actually, the H_Z is the Hamilton operator of an electron orbiting about the origin of the (x,y)-plane in a constant magnetic field $\mathbf{K} = B\partial_z$. It had been established by means of the Maxwell equations.

Mathematical modeling: Zeeman manifolds. An interesting feature of free Zeeman operators, H_Z, is that they can be identified with the Laplace operators of certain Riemannian manifolds, namely, with the Laplacians on two step nilpotent Lie groups endowed with the natural left invariant metrics. The details are as follows.

A 2-step nilpotent metric Lie group is defined on the product $\mathbf{v}\oplus\mathbf{z}$ of Euclidean spaces, where the components, $\mathbf{v}=\mathbb{R}^k$ and $\mathbf{z}=\mathbb{R}^l$, are called X- and Z-space respectively. The main object defining the Lie algebra is the linear space $J_\mathbf{z}$ of skew endomorphisms J_Z, where $Z\in\mathbf{z}$, acting on the X-space. The metric, g, is the left invariant extension of the natural Euclidean metric on the Lie algebra.

Particular 2-step nilpotent Lie groups are the Heisenberg-type Lie groups which are defined by endomorphism spaces satisfying the Clifford condition $J_Z^2 = -|Z|^2 id$. I. e., these metric groups are attached to Clifford modules. In this case the X-space decomposes into the product $\mathbf{v} = (\mathbb{R}^{r(l)})^{a+b} = \mathbb{R}^{r(l)a}\times$

$\mathbb{R}^{r(l)b}$. Endomorphisms J_Z are defined by endomorphisms j_Z acting on the smaller space $\mathbb{R}^{r(l)}$. Namely, the J_Z acts on $\mathbb{R}^{r(l)a}$ resp. $\mathbb{R}^{r(l)b}$ as $j_Z \times \cdots \times j_Z$ resp. $-j_Z \times \cdots \times -j_Z$. The H-type groups are denoted by $H_l^{(a,b)}$, indicating the above decomposition. A point on the nilpotent group is represented by (X, Z). Each group, (N, g), extends into a solvable group (SN, g_s), where a point is represented by (t, X, Z). The solvable extensions of H-type groups are then $SH_l^{(a,b)}$.

The Laplacians on H-type groups are of the form

$$\Delta = \Delta_X + (1 + \frac{1}{4}|X|^2)\Delta_Z + \sum_{\alpha=1}^{r} \partial_\alpha D_\alpha \bullet, \tag{2}$$

where $D_\alpha \bullet$ denotes directional derivatives along the X-fields $J_\alpha(X) = J_{Z_\alpha}(X)$ and endomorphisms J_α are defined by an orthonormal basis $\{Z_\alpha\}$ of the Z-space.

This operator is not the Zeeman operator yet. The Zeeman operator appears on center periodic H-type groups introduced by factorizations, $\Gamma_\gamma \backslash H$, with Z-lattices $\Gamma_\gamma = \{Z_\gamma\}$ of the Z-space. In fact, in this case the L^2 function space is the direct sum of function spaces W_γ spanned by functions of the form $\Psi_\gamma(X, Z) = \psi(X)e^{2\pi \mathbf{i}\langle Z_\gamma, Z\rangle}$. Each W_γ is invariant under the action of the Laplacian, i. e., $\Delta\Psi_\gamma(X, Z) = \Box_\gamma \psi(X)e^{2\pi \mathbf{i}\langle Z_\gamma, Z\rangle}$, where operator $\Box_\gamma$, acting on $L^2(\mathbf{v})$, is of the form

$$\Box_\gamma = \Delta_X + 2\pi \mathbf{i} D_\gamma \bullet -4\pi^2 |Z_\gamma|^2 (1 + \frac{1}{4}|X|^2). \tag{3}$$

Notice that (1) is nothing but (3) on the 3D-Heisenberg group. On a $(k+1)$-dimensional Heisenberg group, defined by a complex structure J acting on the even dimensional Euclidean space $\mathbf{v} = \mathbf{R}^k$, the Laplacian (3) appears in the form

$$\Box_\lambda = \Delta_X + 2\mathbf{i} D_\lambda \bullet -\lambda^2 |X|^2 - 4\lambda^2. \tag{4}$$

Number $k/2$ is interpreted as the number of particles. The single complex structure J is interpreted such that these particles are rotating in the same plane defined by the same constant magnetic field B. In the 2D-version the particles belonging to J and $-J$ are called antiparticles (electrons and positrons). The single parameter λ above is interpreted that the particles are identical up-to the sign of the charge they are loaded.

Operators (1) and (4) are identified by $H_Z = -(1/2)\Box_\lambda$ and by the particular choice $\mu = \hbar = 1, \lambda = eB/2c$ of the constants. Operator (3) contains also the constant $-4\pi^2|Z_\gamma|^2 = 4\lambda^2$, which is proportional to $\hbar^2$ on the microscopic level, thus, it is usually neglected in quantum physics. Also note that the particles described by these Hamiltonians are free ($V = 0$).

The Zeeman operator appears as Laplacian on center periodic 2-step nilpotent Lie groups in a more complex form. These models represent $k/2$ number of charged particles, each of them is orbiting in its own constant magnetic field. The system can be in crystal states represented by the endomorphisms J_γ. The Hamilton operators belonging to these crystal states are $-\frac{1}{2}\Box_\gamma$. This model matches Dirac's famous multi-time theory.

The Riemannian manifolds introduced so far are prototypes of a general Zeeman manifold concept. This general concept is beyond the scope of this talk and will be developed in a subsequent paper. This exposition proceeds with considering 2-step nilpotent Lie groups.

2 Isospectrality constructions

The isospectrality constructions are performed on H-type groups, $H_l^{(a,b)}$, and on their solvable extensions, $SH_l^{(a,b)}$, both with periodic and non-periodic centers. For fixed $a + b$ and l, these groups are defined on the same (X, Z)-resp. (t, X, Z)-space. There is established in many different ways that the local geometries regarding the metrics $g_l^{(a,b)}$ of a family are completely different (cf. the striking examples). Yet, certain submanifolds of these common underlying manifolds have the same Laplace spectra regarding all the metrics of the family. The endomorphism space $J_l^{(a',b')}$ can be constructed by an other endomorphism space, $J_l^{(a,b)}$, of the family such that the irreducible endomorphism spaces, j_l or $-j_l$, by which J_l is expressed are multiplied by -1 on some irreducible subspaces, $\mathbb{R}^{r(l)}$. In retrospect to the physical interpretation, this means that some of the particles are exchanged by their antiparticles. Although this exchange drastically changes the local geometry, the spectrum on the considered domains remains the same.

Constructing the ball$\times$torus- and sphere$\times$torus-examples. These examples are constructed for a family, $\Gamma\backslash H_l^{(a,b)}$, of Z-periodic manifolds defined for fixed l and $a+b$. The submanifolds considered are torus bundles over a ball (resp. sphere) around the origin of the X-space.

To establish an intertwining operator, one should consider, for each invariant space W_α constructed above, an orthogonal transformation on the X-space which conjugate $J_\alpha^{(a,b)}$ to $J_\alpha^{(a',b')}$. This transformation clearly intertwines $\Box_\alpha$ with $\Box'_\alpha$ such that it keeps also the boundary conditions. It induces an appropriate intertwining operator also on the boundary manifolds. The striking examples appear on the quaternionic families $H_3^{(a,b)}$. In this case the sphere$\times$torus-type boundary manifolds in $\Gamma\backslash H_3^{(a+b,0)}$ are homogeneous while the others in the isospectrality family are locally inhomogeneous.

In retrospect, this intertwining operator was constructed in two steps: First, the L^2-function space has been decomposed into invariant subspaces, then, for each invariant subspace an operator intertwining both the Laplacians and the boundary conditions has been found. This operator is derived from a point transformation. For distinct invariant subspaces also these point transformations are distinct. Note that for each invariant space only a single endomorphism J_γ is involved to the constructions.

Constructing the ball- and sphere-type examples. These examples were originally constructed in [Sz2, Sz3]. It turned out just recently that the intertwining operator was incorrectly defined in these articles. The corrected intertwining operator has been introduced in [Sz4]. This correction saved all the previous results and, additionally, it produced new examples defined on sphere$\times$ball- and sphere$\times$sphere-type manifolds. Among them also new striking examples have been found. Next this corrected intertwining operator is constructed.

The ball-type domains are, by definition, diffeomorphic to Euclidean balls such that the sphere-type boundary manifolds are level sets described by equations of the form $\varphi(|X|,|Z|)=0$ resp. $\varphi(|X|,|Z|,t)=0$. These domains are invariant under the action of the orthogonal group $\mathbb{O}(\mathbb{R}^k)\times\mathbb{O}(\mathbb{R}^l)$ (domains of (X,Z)-revolution), which can be visualized such that there is a ball, centered at the origin in the X-space, over the points of which there are Z-balls of radius $R_Z(|X|)$ around the origin in the Z-space. Then the boundary is a level set defined by $\varphi(|X|,|Z|)=|Z|-R_Z(|X|)=0$.

Note that radius $R_Z(|X|)$ is constant along a sphere defined by a constant radius $R_X = |X|$ in the X-space and the ball bundle defined by the Z-balls over this X-sphere is trivial. These are the so called *sphere×ball-type manifolds* whose boundaries are *sphere×sphere-type manifolds*. The new examples, not discussed in the earlier papers, are constructed on these domains and surfaces.

This visualization can be started out by a Z-ball in the Z-space over the points of which there are X-balls of radius $R_X(|Z|)$ given. Then the boundary is defined by $|X| - R_X(|Z|) = 0$. This paper proceeds with the first approach.

In the solvable case one should consider (Z,t)-balls and (Z,t)-spheres around the origin $(0_Z, 1)$. The base manifold is the same X-sphere as before. Note that a Z-ball $B_{R_Z}(0_Z)$ (resp. Z-sphere $S_{R_Z}(0_Z)$) uniquely extends into a geodesic ball (resp. sphere) of the (hyperbolic) (Z,t)-space. A sphere×ball-type domain can be regarded as a level set in a ball-type domain such that the Z-balls (or $(Z,t) - balls$) of the ball type domain are considered over a sphere S_{R_X} in the X-space. Similarly, the sphere×sphere-type manifolds can be regarded as level sets in sphere-type manifolds.

The isospectrality will be investigated in details for the discrete families, $H_l^{(a,b)}$, of Heisenberg type groups defined by the same l and $a+b$. The Laplacian to be considered now is described in (2). Comparing with the Zeeman operator (3), this operator involves all the endomorphisms, making the constructions rather difficult. The Laplacians of the members in a family differ from each other just by the last term, $\mathbf{M}$, which is a compound orbital angular momentum operator. There is explained in [Sz5] that the compound orbital angular momentum operator in Dirac's famous multi-time model, which complies with relativity by endowing each particle in the system with self-time, appears in this form. The spectral investigation both of $\mathbf{M}$ and Δ is completely missing both in physics and mathematics. This explains some extend the difficulties one should face in investigating this Laplacian. Note that this most intriguing operator, $\mathbf{M}$, commutes with both operators in the rest part of (2).

Constructing the intertwining operator. In the *first step* we explicitly describe the *eigenfunctions both of* $\mathbf{M}$ *and* Δ *with no boundary conditions.* Since the $\mathbf{M}$ commutes with the rest part $\mathbf{O}$ of Δ, the eigenfunctions can be sought such that they are eigenfunctions both of $\mathbf{M}$ and $\mathbf{O}$.

In the very first step we look for the eigenfunctions of a single angular momentum operator $\mathbf{D}_V\bullet$, defined for a Z-vector V. For a fixed X-vector Q and the unit Z-vector $V_u = \frac{1}{|V|}V$, consider the X-function $\Theta_Q(X,V_u) = \langle Q+\mathbf{i}J_{V_u}(Q),X\rangle$ and its conjugate $\overline{\Theta}_Q(X,V_u)$. For vector $V=|V|V_u$, these functions are eigenfunctions of $D_V\bullet$ with eigenvalue $-|V|\mathbf{i}$ resp. $|V|\mathbf{i}$. The higher order eigenfunctions are of the form $\Theta_Q^p\overline{\Theta}_Q^q$ with eigenvalue $(q-p)|V|\mathbf{i}$.

In order to find the eigenfunctions of the compound operator $\mathbf{M}$, consider a sphere S_{R_Z} of radius R_Z around the origin in the Z-space. For an appropriate function $\phi(|X|,V)$, depending on $|X|$ and $V\in S_{R_Z}$, define

$$\mathcal{F}_{QpqR_Z}(\phi)(X,Z) = \oint_{S_{R_Z}} e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\Theta_Q^p(X,V_u)\overline{\Theta}_Q^q(X,V_u)dV. \quad (5)$$

By $\mathbf{M}\oint = \oint \mathbf{i}D_V\bullet$, this function is an eigenfunction of $\mathbf{M}$ with the real eigenvalue $(p-q)R_Z$. These functions are eigenfunctions also of Δ_Z with eigenvalue R_Z^2. Also note that these eigenvalues do not change by varying Q.

The function space spanned by functions (5) generated by different ϕ's is not invariant with respect to the action of Δ_X, thus the eigenfunctions of the complete operator Δ do not appear in this form. In order to find the common eigenfunctions, the homogeneous non-harmonic polynomials $\Theta_Q^p\overline{\Theta}_Q^q$ of the X-variable should be exchanged in the above formula for the polynomials $\Pi_X(\Theta_Q^p\overline{\Theta}_Q^q)$, defined by projections, Π_X, onto the space of $(p+q)$-order homogeneous harmonic polynomials of the X-variable. These projections are described in the form $\Pi_X = \Delta_X^0 + B_1|X|^2\Delta_X + B_2|X|^4\Delta_X^2 + \dots$ in [Sz3] (cf. (3.14)). By this formula easily follows that also the functions

$$\mathcal{HF}_{QpqR_Z}(\phi)(X,Z) = \oint_{S_{R_Z}} e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\Pi_X(\Theta_Q^p(X,V_u)\overline{\Theta}_Q^q(X,V_u))dV \quad (6)$$

are eigenfunctions both of $\mathbf{M}$ and Δ_Z with the same eigenvalues as in (5).

When the complete Laplacian (2) is acting on this function, one gets an action which is combination of X-radial differentiation, $\partial_{|X|}$, and multiplications with functions depending just on $|X|$. I. e., the action is completely reduced to X-radial functions. Also this reduced form of the Laplacian is not changing by varying Q. The eigenfunctions of Δ can be found by seeking the eigenfunctions of the reduced operator among the X-radial functions. Since it is not needed, we do not go into further details of this computation here.

The above discussions clearly indicate that functions (5), defined for all S_{R_Z}, can be used to define *operators intertwining the Laplacians* for any two Heisenberg type groups $H_l^{(a,b)}$ and $H_l^{(a',b')}$ satisfying $a+b=a'+b'$. The intertwining operators κ_{Qpq} can be loosely defined by the correspondence

$$\kappa_{Qpq}:\mathcal{F}_{QpqR_Z}(\phi)(X,Z)\to\mathcal{F}'_{QpqR_Z}(\phi)(X,Z), \tag{7}$$

where the associated functions are built up by J_α resp. J'_α in the very same form. For fixed Q and $n,m\in\mathbb{N}$, functions $\mathcal{F}_{QpqR_Z}(\phi)(X,Z)$ defined for $p+q\leq n$, $q-p=m$ and all R_Z span a function space which is closed with respect to the action of Δ. The κ_{Qpq} maps this function space to the corresponding other one by intertwining Δ and Δ' term by term. The intertwining of $\mathbf{M}$ resp. Δ_Z with $\mathbf{M}'$ resp. Δ_Z is clear by the above considerations. One can show that also the Δ_X is intertwined with itself. Moreover, the correspondence $\kappa_{Qpq}:\mathcal{HF}_{QpqR_Z}(\phi)(X,Z)\to\mathcal{HF}'_{QpqR_Z}(\phi)(X,Z)$, defines the same operator as before. Actually, the function spaces of the first type are direct sums of spaces of the second type. The operators defined by the two alternatives agree. The second definition clearly shows the intertwining of Δ_X and Δ'_X.

This was just a loose description of the intertwining operator. One should also keep in mind that it was considered just on an invariant subspace defined by a fixed Q (the global intertwining will be defined, in the end, by splitting the L^2-space into such invariant spaces). For defining this reduced intertwining in a precise form, notice that (5) describes the Fourier transform of a Dirac-type distribution concentrated on the sphere S_{R_Z}. In polar coordinates the Z-Fourier transform can be interpreted as superposition of transforms described in (5). Thus the map defined by the Z-Fourier transform

$$\begin{aligned}\kappa_{Qpq}:\quad & \mathcal{F}_{Qpq}(\phi)=\int_{\mathbb{R}^l}e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\Theta_Q^p(X,V_u)\overline{\Theta}_Q^q(X,V_u)dV\to \\ & \to\mathcal{F}'_{Qpq}(\phi)=\int_{\mathbb{R}^l}e^{\mathbf{i}\langle Z,V\rangle}\phi(|X|,V)\Theta_Q^{\prime p}(X,V_u)\overline{\Theta}_Q^{\prime q}(X,V_u)dV.\end{aligned} \tag{8}$$

is indeed an intertwining operator (cf. also (9)-(10)).

The L^2-function spaces between which this intertwining is defined are denoted by $\mathbf{\Phi}_{Qpq}$ and $\mathbf{\Phi}'_{Qpq}$ respectively. They are the Z-Fourier transforms of the L^2-Hilbert spaces spanned by functions of the form $\phi\Theta_Q^p\overline{\Theta}_Q^q$ and $\phi\Theta_Q^{\prime p}\overline{\Theta}_Q^{\prime q}$, where $\phi(Z)$ is an arbitrary L^2-function. The corresponding function spaces defined by the Fourier transform of functions involving the X-harmonic polynomials $\Pi_X(\Theta^p_Q\overline{\Theta}^q_Q)$ are denoted by Ξ_{Qpq}. They are corresponded to Ξ'_{Qpq} by the $\kappa_{Qpq}:\mathcal{HF}_{Qpq}(\phi)\to\mathcal{HF}'_{Qpq}(\phi)$. (It should not be confusing that this map is denoted by the same formula regarding both the $\mathbf{\Phi}$- and Ξ-function spaces.) Unlike the $\mathbf{\Phi}_{Qpq}$, which is not closed regarding the action of Δ_X, the Ξ_{Qpq} is closed under the action of the complete operator Δ. Exactly this is the primarily reason why these subspaces have been introduced here. Several larger L^2-spaces invariant under the actions of the Laplacians can also be introduced. Invariant space $\Xi_{Q,p+q=n}=\sum_{q=0}^n\Xi_{Q,p=n-q,q}$, defined for a fixed $n\in\mathbb{N}$, will be thoroughly investigated later.

Natural operator, $\omega_{Q\tilde{Q}pq}:\mathbf{\Phi}_{Qpq}\to\mathbf{\Phi}_{\tilde{Q}pq}$ and $\omega_{Q\tilde{Q}pq}:\Xi_{Qpq}\to\Xi_{\tilde{Q}pq}$, intertwining the corresponding function spaces defined by two distinct unit vectors Q and $\tilde{Q}$ can also be introduced. Note that this function spaces are defined on the same H-type group. The existence of these operators will give a clear explanation for the puzzle: How can the spectrum “ignore” the isometries? Since spherical harmonics $\Pi_X(\Theta^p_Q\overline{\Theta}^q_Q)$ are mapped to the same type of harmonics and

$$\mathbf{M}\mathcal{F}_{Qpq}(\phi)=\mathcal{F}_{Qpq}((q-p)|V|\phi)\,,\ \Delta_Z\mathcal{F}_{Qpq}(\phi)=\mathcal{F}_{Qpq}(-|V|^2\phi),\quad (9)$$

$$\mathbf{M}\mathcal{HF}_{Qpq}(\phi)=\mathcal{HF}_{Qpq}((q-p)|V|\phi)\,,\ \Delta_Z\mathcal{HF}_{Qpq}(\phi)=\mathcal{HF}_{Qpq}(-|V|^2\phi),\ (10)$$

all these operators obviously intertwine the corresponding Laplacians. The second line follows from the first one by the commutativity of operator $\mathbf{D}_V\bullet$ with the projection Π_X. Also note that these formulas prove the intertwining property also for κ_{Qpq}, since the same formulas are valid also for an other group $H_l^{(a',b')}$, where $a'+b'=a+b$, in the family.

By these formulas, both the κ_Q- and the $\omega_{Q\tilde{Q}}$-intertwining operators can be derived from point transformations of the form (K_Q,id_Z) resp. $(O_{Q\tilde{Q}},id_Z)$, where id_Z is the identity on the Z-space and the first operators are orthogonal transformations on the X-space. The latter ones transform the subspace S_Q, spanned by Q and all $J_{Z_0}(Q)$, to S'_Q resp. $S_{\tilde{Q}}$. This part of the transformation is uniquely determined, whereas, between the complement spaces it can be arbitrary orthogonal transformation.

This observation easily proves that the *Dirichlet condition* is intertwined by each of the above operators on each invariant subspaces considered above. The same statement can be proved also for the Neumann conditions.

The isospectrality on ball- and sphere$\times$ball-type manifolds of nilpotent groups can be established in two different ways. In the first one the $L^2_{\mathbf{C}}$-Hilbert space is decomposed into direct sum of invariant subspaces described above, then, the κ-type intertwining operators on these subspaces define global intertwining operator. While in the periodic case there is only one reasonable decomposition, compelled by the Fourier-Weierstrass decomposition defined by the Z-lattice Γ, in this case this splitting has to be constructed. Different splittings define different intertwining operators.

To make the first step in this decomposition, note that the whole L^2 function space is spanned by the Δ-invariant sub-spaces Ξ_{Qpq}, considered for all Q, p, and q. This statement can be established by proving that the invariant space $\Xi_{n=p+q} = \sum_Q \Xi_{Qn=p+q}$, defined for fixed $n = p+q$, is nothing but the space spanned by the functions of the form $\phi(|X|)H^{(n)}(X)\varphi(Z)$, where $H^{(n)}$ is an n^{th}-order homogeneous harmonic polynomial and φ is an L^2-function. This space is spanned by the invariant spaces $\Xi_{pq} = \sum_Q \Xi_{Qpq}$ defined for fixed p and q satisfying $p+q=n$. The number of independent Q's such that the corresponding subspaces span this latter total space is

$$d_{pq} = \binom{p+k-1}{k-1}\binom{q+k-1}{k-1}, \tag{11}$$

where k is the dimension of the X-space. Furthermore, one can find $d_n = \binom{n+k-1}{k-1}$ number of independent vectors, $Q_1, \dots, Q_{d_n}$, such that for any fixed p and q the Ξ_{pq} is the direct sum of subspaces $\{\Xi_{Q_1pq}, \dots, \Xi_{Q_{d_{pq}}pq}\}$. By considering all p and q satisfying $p+q=n$, one has a decomposition for $\Xi_{n=p+q}$. One can also prove that a function $f \in \Xi_{n=p+q}$ satisfies a given boundary condition if and only if all of the component functions in the decomposition of f with respect to this direct sum splitting satisfy the boundary condition. Thus, by intertwining with each κ_{Q_ipq}, one has a global intertwining between $\Xi_{n=p+q}$ and $\Xi'_{n=p+q}$ which intertwines both the Laplacians and the boundary conditions.

In the other isospectrality proof the intertwining operators $\omega_{Q\tilde{Q}pq}$ are applied. The elements, $\{\lambda_{n=p+q,i}\}$, of the spectrum appear on each $\Xi_{Q,n=p+q}$ with multiplicity, say $m_{n=p+q,i}$. Then the multiplicity regarding the whole L^2-space can be determined by the dimensions described above. On the other hand, for $Q \in \mathbb{R}^{r(l)a}$, the isospectrality obviously follows from $\Xi_{Qpq} = \Xi'_{Qpq}$. This proves the desired isospectrality completely.

The latter proof clearly demonstrates that the spectrum can really be "ignorant" of the isometries. In fact, there is a subgroup, $\mathbf{Sp}(a)\times\mathbf{Sp}(b)$, of isometries on a Heisenberg-type group $H_3^{(a,b)}$ which acts as the identity on the Z-space. Note that these isometries act transitively on the X-spheres of $H_3^{(a+b,0)}$, implying the intertwining property for $\omega_{Q\tilde{Q},n=p+q}$. Though the isometries are not transitive on the X-spheres of the other members of the isospectrality family, yet, the $\omega_{Q\tilde{Q},n=p+q}$ is still an intertwining operator on its own, without the help of the isometries.

Intertwining on the boundaries. Since they intertwine the Dirichlet boundary condition, the above intertwining operators induce, by restrictions, maps of functions defined on the boundary manifolds. It turns out that these maps are indeed intertwining operators on the boundary manifolds, establishing the isospectrality there too. Then, one has new striking examples on sphere$\times$sphere-type manifolds. In fact, the complete **group of isometries** on sphere$\times$sphere-type manifolds of $H_3^{(a,b)}$ is the semi direct product $(\mathbf{Sp}(a)\times\mathbf{Sp}(b))\cdot SO(3)$, where the action of $SO(3)$, represented by the unit quaternionic numbers, q, is defined by $(X_1,\dots,X_{a+b},Z)\to(qX_1\overline{q},\dots,qX_{a+b}\overline{q},qZ\overline{q})$. This complete group acts transitively on the sphere-$\times$sphere-type manifolds of $H_3^{(a+b,0)}$, while the other manifolds in the isospectrality family are locally inhomogeneous. Also the dimensions of the isometry groups belonging to the members of the family are distinct.

The isospectrality theorem naturally extends to the **solvable extensions**. The Laplacians on the ambient- and boundary-manifolds, furthermore, the normal vectors to the boundaries are described in formulas (1.12), (3.30), and (3.29) of [Sz3]. The generator functions are of the form $\phi(|X|,t,V)$ in this case, i. e., the intertwining is led back to the nilpotent group. The group of isometries acting on the sphere$\times$sphere-type manifolds of $SH_3^{(a,b)}$, where $ab\neq 0$, is $(\mathbf{Sp}(a)\times\mathbf{Sp}(b))\cdot SO(3)$, while on $SH_3^{a+b,0)}$ it is $\mathbf{Sp}(a+b)\cdot\mathbf{Sp}(1)$. As in the nilpotent case, this yields the same statement regarding the homogeneity property of the manifolds also in the solvable isospectrality family. Thus we have

Theorem 1. *The ball- and sphere-type manifolds, determined by the same function $\varphi(|X|,|Z|)$ on the members of an isospectrality family $H_l^{(a,b)}$, are isospectral. On ball-type manifolds this statement is proved by constructing, first, intertwining operators $\kappa_{Qpq}:\mathbf{\Xi}_{Qpq}\to\mathbf{\Xi}'_{Qpq}$ between Laplace-invariant subspaces, then, by choosing a complete independent subspace-system, a global* *intertwining between the L^2-spaces is established. In an other proof operators $\omega_{Q\tilde{Q}pq} : \mathbf{\Xi}_{Qpq} \to \mathbf{\Xi}_{\tilde{Q}pq}$, associating the functions of the same H-type group to each other, are constructed. This proof better reveals how does the spectrum "ignore" the isometry group. On the sphere-type boundary manifolds the isospectralities are established by double sided intertwining operators.*

This isospectrality statement extends to σ-equivalent 2-step nilpotent groups defined by totally anticommutative endomorphism spaces as well as to the ball- and sphere-type manifolds of the solvable extensions of these groups. These examples include the striking examples, constructed on the geodesics spheres of $SH_3^{(a,b)}$, where one of the family-members is homogeneous while the others are locally inhomogeneous.

New examples are constructed on sphere$\times$ball- and sphere$\times$sphere-type manifolds. Among them two are particularly interesting. In fact, the isospectrality family of sphere$\times$sphere-type manifolds, constructed both on $H_3^{(a,b)}$ and $SH_3^{(a,b)}$, the metric is homogeneous for the manifold belonging to the pair $(a+b,0)$ or $(0,a+b)$, while the metrics satisfying $ab \neq 0$ are locally inhomogeneous. Also the dimensions of groups of isometries acting on the members are different.

Generalizations. General 2-step nilpotent Lie groups are defined by general endomorphism spaces, $J_{\mathbf{z}}$, of skew endomorphisms acting on the X-space. A σ-deformation of the endomorphism space is defined by an orthogonal involutive transformation σ on the X-space which commutes with all endomorphisms from $J_{\mathbf{z}}$. The σ-deformed endomorphism space consists of endomorphisms σJ_Z. Note that the family of Cliffordian endomorphism spaces $J_l^{(a,b)}$ defined by the same $a+b$ and l consists of σ-equivalent endomorphism spaces. In papers [Sz2, Sz3] the isospectrality is stated on the ball- and sphere-type domains of such 2-step nilpotent Lie groups and their solvable extensions whose endomorphism spaces contain at least one anticommutator. However, the above isospectrality constructions extend to the ball-, sphere-, sphere$\times$ball-, and sphere$\times$sphere-type manifolds of two-step nilpotent Lie groups and their solvable extensions which are defined by σ-equivalent endomorphism spaces. This extension is non-trivial. The above ideas work out just for H-type groups and their solvable extensions. The extensions require substantial modifications both in the formulas defining the intertwining operators and in investigating the boundary conditions.

3 Normal de Broglie Geometry

Infinities in Quantum Electrodynamics (QED). The problem of infinities (divergent integrals), which is present in calculations since the early days of quantum field theory (Heisenberg-Pauli (1929-30)) or elementary particle physics (Oppenheimer (1930) and Waller (1930) in electron theory), is treated by *renormalization* in the current theories. This perturbative tool provides the desired finite quantities by differences of infinities. This problem is the legacy of controversial concepts such as *point mass* and *point charge* of classical electron theory, which provided the first warning that a point electron will have infinite electromagnetic self-mass: the mass $e^2/6\pi ac^2$ for a surface distribution of charge with radius a blows up for $a \to 0$. In quantum field theory the Hamiltonian of the field is proportional to this electromagnetic self-mass. This is why this infinity launched one of the deepest crisis's in the history of physics.

The infinities, related to the divergence of the summations over all possible distributions of energy/momentum of the virtual particles, mostly appear in the form of infinite traces of kernels such as the Wiener-Kac kernel e^{-tH} or the Dirac-Feynman kernel $e^{-tH\mathbf{i}}$. The WK-kernel provides the fundamental solution of the heat equation while the DF-kernel provides the fundamental solution of the Schrödinger equation. The infinite trace assigns infinite measures to physical objects such as self-mass, self-charge, e.t.c.. Because of the divergent integrals appearing in its construction, also the Feynman measure, which is analogous to the well defined Wiener-Kac measure on the path-spaces, requires renormalization.

This paper offers a new non-perturbative approach to this problem. The main idea in this approach can be briefly described as follows. In the first step the quantum Hilbert space $\mathcal{H}$ (on which the quantum Hamilton operator H is acting) is decomposed into the direct sum of H-invariant subspaces, called Zeeman zones. Then all the operator-actions, such as the the heat- or Feynman-flows, are considered on these invariant subspaces separately. It turns out that both the Wiener-Kac and Dirac-Feynman kernels are of the trace class on each zone, furthermore, both define the corresponding zonal measures on the path-spaces rigorously.

Introducing the zones. The Hilbert space, $\mathcal{H}$, whose zonal decomposition is to be established is the space of complex valued L^2-functions defined on the X-space. It is isomorphic to the weighted space defined by the Gauss density $d\eta_\lambda(X) = e^{-\lambda|X|^2}dX$, which is spanned by the complex valued polynomials.

Next the Hilbert space is interpreted in this way. The natural *complex Heisenberg group representation* on $\mathcal{H}$ is defined by

$$\rho_{\mathbf{c}}(z_i)(\psi) = (-\partial_{\overline{z}_i} + \lambda z_i\cdot)\psi \quad , \quad \rho_{\mathbf{c}}(\overline{z}_i)(\psi) = \partial_{z_i}\psi, \tag{12}$$

where $\{z_i\}$ is a complex coordinate system on the X-space. This representation is reducible. In fact, it is irreducible on the space of holomorphic functions, where it is called Fock representation. Besides the holomorphic subspace there are infinitely many other irreducible invariant subspaces. In the literature only the Fock representation, defined on the space of holomorphic functions, is well known. The above representation is called *extended Fock representation.* In the function operator correspondence, this representation associates operator (1) to the Hamilton function of an electron orbiting in constant magnetic field.

The zones are defined in two different ways. First, they can be defined by the invariant subspaces of representation (12). The actual construction uses Gram-Schmidt orthogonalization. On the complex plane $\mathbf{v} = \mathbf{C}$, which corresponds to the 2D-Zeeman operator (1), the first zone, $\mathcal{H}^{(0)}$, is the holomorphic zone spanned by the holomorphic polynomials. To construct the second zone one considers the function space $G^{(1)}$ consisting of functions of the form $\overline{z}h$, where the h is a holomorphic polynomial. Then $\mathcal{H}^{(1)}$ is the orthogonal component of $G^{(1)}$ to the previous holomorphic zone. E. t. c., one has all zones, $\mathcal{H}^{(a)}_\lambda$, by continuing with the Gram-Schmidt orthogonalization applied to function spaces $G^{(a)}$, which are spanned by functions of the form $\overline{z}^a h$. The zone index a indicates the maximal number of the antiholomorphic coordinates $\overline{z}$ in the polynomials spanning the zone.

In the 2D-case all the zones are irreducible under the action of the extended Fock representation. In the higher dimensions, however, the Gram-Schmidt process results reducible zones, called *gross zones.* More precisely, the holomorphic zone is always irreducible and the gross zones of higher indexes decompose into *irreducible zones* which can also be explicitly described.

The second technique defines the very same zones by computing the spectrum and the corresponding eigenfunctions explicitly. According to these computations, the eigenfunctions appear in the form

$$h^{(p,v)}(X) = H^{(p,v)}(X)e^{-\lambda|X|^2/2} \tag{13}$$

with the corresponding eigenvalues

$$-((4p+k)\lambda + 4k\lambda^2), \tag{14}$$

where p resp. v are the holomorphic resp. antiholomorphic degrees of polynomial $H^{(p,v)}$. Numbers $l = p + v$ and $m = 2p - l$ are called azimuthal and magnetic quantum numbers respectively. The above function is an eigenfunction also of the magnetic dipole moment operator with eigenvalue m.

In this new way the zones are created such that the above eigenfunction falls into the zone with index v. According to the formula $v = \frac{1}{2}(l-m)$, the zones are determined by the magnetic quantum number m. Thus a zone exhibits the magnetic state of the particle.

Note that eigenvalues (14) are independent of the antiholomorphic index and they depend just on the holomorphic index. As a result, each eigenvalue has infinite multiplicity. On the irreducible zones, however, each multiplicity is $k/2$. (Here we suppose that there is only one parameter λ involved, meaning that the particles are identical. If the particles (i. e., the λ_i's) are properly distinct, then the multiplicity is 1 on each zone.) Moreover, two irreducible zones are isospectral.

Introducing the point-spreads by projection kernels. It is remarkable that all the important objects such as the projections onto the zones, the zonal heat- and Feynman-kernels and their well defined trace, the zonal partition functions (defined with no regularization), and the several Feynman-Kac type formulas can explicitly be computed [Sz5, Sz6].

First the *projection operators*, $\delta^{(a)}$, onto the zones $\mathcal{H}^{(a)}$ are established. If $\{\varphi_i^{(a)}\}_{i=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}^{(a)}$, then the corresponding projection can be formally defined as convolution with the kernel

$$\delta^{(a)}(z,w) = \sum_i \varphi_i^{(a)}(z)\overline{\varphi_i^{(a)}(w)}, \tag{15}$$

where z and w represent complex vectors on $\mathbf{C}^{\frac{k}{2}} = \mathbf{R}^k$. Interestingly enough, these self-adjoint operators are integral operators having a smooth Hermitian integral kernels. These kernels can be interpreted as restrictions of the global Dirac delta distribution, $\delta_z(w) = \sum \varphi_i(z)\overline{\varphi}_i(w)$, onto the zones. They have the following explicit form

$$\delta^{(a)}_{\lambda z}(w) = \frac{\lambda^{k/2}}{\pi^{k/2}} L_a^{((k/2)-1)}(\lambda|z-w|^2)e^{\lambda(z\cdot\overline{w}-\frac{1}{2}(|z|^2+|w|^2)}, \tag{16}$$

where $L_a^{((k/2)-1)}(t)$ is the corresponding Laguerre polynomial. Among these kernels only the first one, the projection kernel onto the holomorphic zone, is well known. It is nothing but the Bergman kernel. The new mathematical feature of these formulas is that they are explicitly determined regarding each zone and not just for the holomorphic zone.

These kernels represent one of the most important concepts in this theory. They can be interpreted such that, on a zone, a point particle appears as a spread described by the above wave-kernel. Note that how these kernels, called zonal point-spreads, are derived from the one defined for the holomorphic zone. This holomorphic spread is just multiplied by the radial Laguerre polynomial corresponding to the zone. These point-spreads show the most definite similarity to the de Broglie waves packets (cf. [Bo], pages 61), suggesting that a point particle concentrated at a point Z appears on a zone as an object spread around Z as a wave-package. The wave-function is described by the above kernel explicitly.

The wave-package interpretation of physical objects started out with the de Broglie theory. This concept was finalized in the Schrödinger equation. The mathematical formalism did not follow this development, however, and the Schrödinger theory is built up on a mathematical background not excluding the existence of the controversial point objects. On the contrary, an electron must be considered as a point-object in the Schrödinger theory as well (cf. Weisskopf's argument on this problem in [Schw, Sz5]). An other demonstration for the presence of point particles in classical theory is the duality principle, stating that objects manifest themselves sometimes as waves and sometimes as point particles. The bridge between the two visualizations is built up in Born's probabilistic theory, where the probability for that that a particle, attached to a wave ξ, can be found on a domain D is measured by $\int_D \xi\overline{\xi}$.

These controversial point-objects, by having infinite self-mass or self-charge attributed to them by the Schrödinger equation, launched one of the deepest crisis's in the history of physics. In the zonal theory de Broglie's idea is established on a mathematical level. Although the points are ostracized from this theory, the point-spreads still bear some reminiscence of the point-particles. For instance, they are the most compressed wave-packages and all the other wave-functions in the zone can be expressed as a unique superposition of the point-spreads. If ξ is a zone-function, the above integral measures the probability that the center of a point-spread is on the domain D. This interpretation restores, in some extend, the duality principle in the zonal theory.

Function $\delta^{(a)}_{\lambda Z}\overline{\delta}^{(a)}_{\lambda Z}$ is called the density of the spread around Z. By this reason, function $\delta^{(a)}_{\lambda Z}$ is called spread-amplitude. Both the spread-amplitude and spread-density generate well defined measures on the path-space. This space consists of continuous curves connecting two arbitrary points. Both measures can be constructed by the method applied in constructing the Wiener measure.

The point-spread concept bears some remote reminiscence of Heisenberg's suggestion (1938) for the existence of a fundamental length L, analogously to h, such that field theory was valid only for distances larger than L and so divergent integrals would be cut off at that distance. This idea has never became an effective theory, however. Other distant relatives of the point-spread-concept are the smeared operators, i. e. those suitably averaged over small regions of space-time, considered by Bohr and Rosenfeld in quantum field theory. There are also other theories where an electron is considered to be extended. Most of them fail on lacking the explanation for the question: Why does an extended electron not blow up? The zonal theory is checked against this problem in [Sz5], section (F) "Linking to the blackbody radiation; Solid zonal particles".

Global Wiener-Kac and Feynman flows. Both definitions imply that the zones are invariant under the action of the Hamilton operator, therefore, kernels such as the *heat (Wiener-Kac) and Feynman-Dirac kernels* can be restricted onto them. The zonal kernels are defined by these restrictions.

Since the spectrum is discrete, also the global kernels, defined for the total space $\mathcal{H}$, can be introduced by the trace formula using an orthonormal basis consisting of eigenfunctions on the whole space $\mathcal{H}$. Despite of the infinite multiplicities on the global setting, both global kernels are well defined smooth functions. If the Zeeman operator H_Z is non-degenerated such that the distinct non-zero parameters $\{\lambda_i\}$, $i = 1, \dots, r$, are defined on k_i-dimensional subspaces, then for the Wiener-Kac kernel we have

$$d_{1\gamma}(t, X, Y) = e^{-tH_Z}(t, X, Y) = \tag{17}$$
$$= \prod \big(\frac{\lambda_i}{2\pi sinh(\lambda_i t)}\big)^{k_i/2} e^{-\sum \lambda_i(\frac{1}{2}coth(\lambda_i t)|X_i - Y_i|^2 + \mathbf{i}\langle X_i, J(Y_i)\rangle}.$$

This kernel satisfies the Chapman-Kolmogorov identity as well as the limit property $\lim_{t\to 0_+} d_1(t, X, Y) = \delta(X, Y)$, however, it is not of the trace class. Thus functions such as the partition function or the zeta function are not defined in the standard way. Note that by regularization (renormalization) only well defined relative(!) partition and zeta functions are introduced.

The explicit form of the global Feynman-Dirac kernel is

$$d_{\mathbf{i}}(t, X, Y) = e^{-t\mathbf{i}H_Z}(t, X, Y) = \tag{18}$$
$$= \prod \big(\frac{\lambda_i}{2\pi \mathbf{i} sin(\lambda_i t)}\big)^{k_i/2} e^{\mathbf{i}\sum \lambda_i\{\frac{1}{2}cot(\lambda_i t)|X_i - Y_i|^2 - \langle X_i, J(Y_i)\rangle\}}.$$

Since for fixed t and X function $d_{\mathbf{i}}(t, X, Y)$ is neither L^1- nor L^2-function of the variable Y, the integral required for the Chapman-Kolmogorov identity is not defined for this kernel. Neither is it of the trace class. Nevertheless, it satisfies the above limit property. Thus the constructions with the Feynman kernel lead to divergent integrals in the very first step.

It is well known in the history that Kac, who tried to understand Feynman, was able to introduce a rigorously defined measure on the path-spaces only by the kernel e^{-tH}. This measure was, actually, established earlier by Wiener for the Euclidean Laplacian Δ_X. Note that the heat kernel involves a Gauss density which makes this constructions possible. Whereas, the Feynman-Dirac kernel does not involve such term. This is why no well defined constructions can be carried out with this kernel. One can strait out all this difficulties, however, by considering these constructions on the zones separately.

Zonal Wiener-Kac and Feynman flows. Also the zonal WK- resp. FD-kernels are well defined smooth functions. The gross **zonal Wiener-Kac kernels** are of the trace class, which can be described, along with their

partition functions, by the following explicit formulas.

$$d_1^{(0)}(t,X,Y)=\prod\big(\frac{\lambda_ie^{-\lambda_it}}{\pi}\big)^{\frac{k_i}{2}}e^{\sum\lambda_i(-\frac{1}{2}(|X_i|^2+|Y_i|^2)+e^{-2\lambda_it}\langle X_i,Y_i+\mathbf{i}J(Y_i)\rangle)},\quad(19)$$

$$d_1^{(a)}(t,X,Y)=(L_a^{(\frac{k}{2}-1)}(\sum\lambda_i|X_i-Y_i|^2)+LT_1^{(a)}(t,X,Y))d_1^{(0)}(t,X,Y),\quad(20)$$

where $LT_1^{(1)}$ is of the form

$$\begin{gathered}LT_1^{(1)}(t,X,Y))=(1-e^{-2t})lt_1^{(1)}(t,X,Y))=\\(1-e^{-2t})(|X|^2+|Y|^2-1-(1+e^{-2t})\langle X,Y+\mathbf{i}J(Y)\rangle)\end{gathered}\quad(21)$$

and for the general terms, $LT_1^{(a)}$, recursion formula can be established. Furthermore,

$$\mathcal{Z}_1^{(a)}(t)=Trd_1^{(a)}(t)=\binom{a+(k/2)-1}{a}\prod\frac{e^{-\frac{k_i\lambda_it}{2}}}{(1-e^{-2\lambda_it})^{\frac{k_i}{2}}}=TrD_1^{(a)}(t),\quad(22)$$

where $D_1^{(a)}(t,X,Y)=L_a^{(\frac{k}{2}-1)}(\sum\lambda_i|X_i-Y_i|^2)d_1^{(0)}(t,X,Y)$ is the dominant zonal kernel. The remaining long term kernel in the WK-kernel vanishes for $\lim_{t\to 0_+}$ and is of the 0 trace class. The zonal WK-kernels satisfy the Chapman-Kolmogorov identity along with the limit property $\lim_{t\to 0_+}d_1^{(a)}=\delta^{(a)}$.

Similar statements can be established regarding the zonal DF-flow. The gross **zonal Dirac-Feynman kernels** are of the trace class which, together with their **partition functions**, can be described by the following explicit formulas.

$$d_{\mathbf{i}}^{(0)}(t,X,Y)=\prod\big(\frac{\lambda_ie^{-\lambda_it\mathbf{i}}}{\pi}\big)^{\frac{k_i}{2}}e^{\sum\lambda_i(-\frac{1}{2}(|X_i|^2+|Y_i|^2)+e^{-2\lambda_it\mathbf{i}}\langle X_i,Y_i+\mathbf{i}J(Y_i)\rangle)},\quad(23)$$

$$d_{\mathbf{i}}^{(a)}(t,X,Y)=(L_a^{(\frac{k}{2}-1)}(\sum\lambda_i|X_i-Y_i|^2)+LT_{\mathbf{i}}^{(1)}(t,X,Y))d_{\mathbf{i}}^{(0)}(t,X,Y),\quad(24)$$

where $LT_{\mathbf{i}}^{(1)}$ is described by

$$\begin{gathered}LT_{\mathbf{i}}^{(1)}(t,X,Y))=(1-e^{-2t\mathbf{i}})lt_{\mathbf{i}}^{(1)}(t,X,Y))=\\(1-e^{-2t\mathbf{i}})(|X|^2+|Y|^2-1-(1+e^{-2t\mathbf{i}})\langle X,Y+\mathbf{i}J(Y)\rangle).\end{gathered}\quad(25)$$

and a general long term, $LT_{\mathbf{i}}^{(a)}$, can be defined recursively. Furthermore,

$$\mathcal{Z}_{\mathbf{i}}^{(a)}(t) = Trd_{\mathbf{i}}^{(a)}(t) = \binom{a+(k/2)-1}{a} \prod \frac{e^{-\frac{k_i\lambda_i t\mathbf{i}}{2}}}{(1-e^{-2\lambda_i t\mathbf{i}})^{\frac{k_i}{2}}} = TrD_{\mathbf{i}}^{(a)}(t), \quad (26)$$

where $D_{\mathbf{i}}^{(a)}(t,X,Y) = L_a^{(\frac{k}{2}-1)}(\sum \lambda_i |X_i - Y_i|^2) d_{\mathbf{i}}^{(0)}(t,X,Y)$ is the dominant kernel. The remaining longterm term in the zonal DF-kernel is of the 0 trace class.

The zonal DF-kernels are zonal fundamental solutions of the Schrödinger equation $(\partial_t + \mathbf{i}(H_Z)_X) d_{\mathbf{i}\gamma}^{(a)}(t,X,Y) = 0$, satisfying the Chapman-Kolmogorov identity as well as the limit property $\lim_{t\to 0_+} d_{\mathbf{i}}^{(a)} = \delta^{(a)}$.

On the zones the WK- and the FD-kernels are not just of the trace class. They both define, rigorously, complex zonal measures, the **zonal Wiener-Kac measure** $dw_{1xy}^{T(a)}(\omega)$ and the **zonal Feynman measure** $dw_{\mathbf{i}xy}^{T(a)}(\omega)$, on the space of continuous curves $\omega : [0,T] \to \mathbb{R}^k$ connecting two points x and y. The existence of zonal WK-measure is not surprising, since this measure exists even for the global setting. However, the trace class property is a new feature, indeed. In case of the zonal Feynman measure both the trace class property and the existence of rigorous zonal Feynman measures are new features. Note that also the zonal DF-kernels involve a Gauss kernel which makes these constructions well defined.

References

[AB] Y. Aharanov and D. Bohm: Significance of electromagnetic potentials in the quantum theory. *Phys. Rev.*, 115:485–491, 1959.

[Bo] D. Bohm: Quantum Theory, Dover, 1979.

[F] R. P. Feynman: QED. Princeton Univ. Press, 1988.

[FH] R. P. Feynman, A. R. Hibbs: Quantum Mechanics and Path Integrals. McGraw-Hill, 1965.

[Ka] A. Kaplan: Riemannian nilmanifolds attached to Clifford modules. *Geom. Dedicata*, 11:127–136, 1981.

[Sch] S. S. Schweber: QED and the Man Who Made It: Dyson, Feynman, Schwinger, and Tomonaga. Princeton Univ. Press, 1994.

[Schw] J. Schwinger (Editor): Selected Papers on Quantum Electrodinamics. Dover Publ. INC, 1958.

[Sz1] Z. I. Szabó: Locally non-isometric yet super isospectral spaces. *Geom. funct. anal. (GAFA)*, 9:185–214, 1999.

[Sz2] Z. I. Szabó: Isospectral pairs of metrics on balls, spheres, and other manifolds with different local geometries. *Ann. of Math.*, 154:437–475, 2001.

[Sz3] Z. I. Szabó: A cornucopia of isospectral pairs of metrics on spheres with different local geometries. *Ann. of Math.*, 161:343–395, 2005.

[Sz4] Z. I. Szabó: Correction and improvement added to "Isospectral pairs of metrics on balls and spheres with different local geometries". *DG/0510202 (submitted)*

[Sz5] Z. I. Szabó: Theory of zones on Zeeman manifolds: A new approach to the infinities of QED. *DG/0510660 (submitted)*

[Sz6] Z. I. Szabó: Pauli-Dirac operators and anomalous zones on Zeeman manifolds. submitted.

[To] S. Tomonaga: The Story of Spin. Univ. of Chicago Press, 1997.

[Ton1] A. Tonomura, N. T. Matsuda, R. Suzuki, A. Fukuhara, N. Osakabe, H. Umezaki, J. Endo, K. Shinagawa, Y. Sugita, and H. Fujiwara: Observation of Aharanov-Bohm effect by electron holography. *Phys. Rev. Lett.*, 48:1443–1446, 1982.

[Ton2] A. Tonomura, N. Osakabe, T. Matsuba, T. Kawasaki, J. Endo, S. Yano, and H. Yamada: Evidence for Aharanov-Bohm effect with magnetic field completely shielded from electron wave. *Phys. Rev. Lett.*, 56:792-795, 1986.

[V] M. Veltman: Facts and Mysteries in Elementary Particle Physics. World Scientific, 2003.