EQUIDISTRIBUTION AND GENERALIZED MAHLER MEASURES

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Abstract. If K is a number field and $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ is a rational map of degree $d > 1$, then at each place v of K, one can associate to φ a generalized Mahler measure for polynomials $F \in K[t]$. These Mahler measures give rise to a formula for the canonical height $h_\varphi(\beta)$ of an element $\beta \in \overline{K}$; this formula generalizes Mahler's formula for the usual Weil height $h(\beta)$. In this paper, we use diophantine approximation to show that the generalized Mahler measure of a polynomial F at a place v can be computed by averaging $\log |F|_v$ over the periodic points of φ.

The usual Weil height of a rational number x/y, where x and y are integers without a common prime factor, is defined as $h(x/y) = \max(|x|, |y|)$. More generally, one can define the usual Weil height $h(\beta)$ of an algebraic number β in a number field K by summing over all of the absolute values of K. Mahler ([Mah60]) has proven that if F is a nonzero irreducible polynomial in $\mathbb{Z}[t]$ with coprime coefficients such that $F(\beta) = 0$, then

$$\deg(F)h(\beta) = \int_0^1 \log |F(e^{2\pi i\theta})| d\theta. \tag{0.0.1}$$

The quantity $\int_0^1 \log |F(e^{2\pi i\theta})| d\theta$ is often referred to as the *Mahler measure* of F.

It is easy to see that $h(\beta^2) = 2h(\beta)$ for any algebraic number β. Similarly, it is easy to check that for any continuous function g on the unit circle, we have

$$\int_0^1 g((e^{2\pi i\theta})^2) d\theta = \int_0^1 g(e^{2\pi i\theta}) d\theta.$$

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Furthermore, the unit circle is the Julia set of φ. Thus, Mahler's formula says that one obtains the height of an algebraic number by integrating its minimal polynomial against the invariant measure for φ on the Julia set of φ.

Now, let $\varphi : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ be any nonconstant rational map. Brolin ([Bro65]) and Lyubich ([Lyu83]) have constructed a φ-invariant probability measure μ_φ with support on the Julia set of φ; Freire, Lopes, and Mañe ([FLM83]) have demonstrated that this measure is the *unique* φ-invariant probability measure μ_φ with support on the Julia set of φ. When φ is defined over a number field K, Call and Silverman ([CS93]) have constructed a height function h_φ with the properties that: (1) $h_\varphi(\varphi(x)) = (\deg \varphi) h_\varphi(x)$ and (2) there is a constant C_φ such that$|h(x) - h_\varphi(x)| < C_\varphi$ for all $x \in \mathbb{P}^1(\overline{K})$. In [PST04], it is shown that Mahler's formula (0.0.1) generalizes to the adelic formula

$$(\deg F) h_\varphi(x) = \sum_{\text{places } v \text{ of } K} \int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v \, d\mu_{\varphi,v},$$

where β is an algebraic point, F is a nonzero irreducible polynomial in $\mathbb{Q}[t]$ such that $F(\beta) = 0$, the measure $\mu_{\varphi,v}$ at an archimedean place is the φ-invariant probability measure constructed by Brolin and Lyubich, and the integral $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v \, d\mu_{\varphi,v}$ at a finite place v is defined analogously to the integrals at the archimedean places.

Lyubich [Lyu83] has also proven that for any continuous function g and any archimedean place v, the integrals $\int_{\mathbb{P}^1(\mathbb{C}_v)} g \, d\mu_{\varphi,v}$ can be computed by averaging g on the periodic points of φ; that is to say,

$$\lim_{k \to \infty} \frac{1}{d^k} \sum_{\varphi^k(w) = w} g(w) = \int_{\mathbb{P}^1(\mathbb{C}_v)} g \, d\mu_{\varphi,v}. \tag{0.0.2}$$

Autissier ([Aut01]), Bilu ([Bil97]), Szpiro, Ullmo, and Zhang ([SUZ97]), and others have obtained generalizations and variations of this result. The most recent generalization, proven independently by Baker and Rumely ([BR05]), Chambert-Loir ([CL04]), and Favre and Rivera-Letelier ([FRL04b] and [FRL04a]) states that (0.0.2) continues to hold when the periodic points w such that $\varphi^k(w) = w$ are replaced by the conjugates of any infinite nonrepeating sequence of algebraic points with height tending to 0 and when the measure $\mu_{\varphi,v}$ is a φ-invariant measure on the v-adic Berkovich space (see [Ber90]) for a finite place v.

The function $\log |F|$, for F a polynomial, is not continuous, of course. Thus, the equidistribution results cited above do not allow us to compute Mahler measures by averaging $\log |F|$ over points of small height.

One can, however, show that for any $\beta \in \bar{\mathbb{Q}}$, we have

$$[\mathbb{Q}(\beta):\mathbb{Q}]h(\beta) = \lim_{k\to\infty} \frac{1}{d^k} \sum_{\xi^n=1} \log|F(\xi)| = \int_0^1 \log|F(e^{2\pi i\theta})|d\theta, \tag{0.0.3}$$

where F is a nonzero irreducible polynomial in $\mathbb{Z}[t]$ with coprime coefficients such that $F(\beta) = 0$ (see [EW99, Chapter 1], [Sch74]). Everest, Ward, and Ní Fhlathúin have proved similar results for maps that come from multiplication on an elliptic curve ([EW99, Chapter 6], [EF96]). The proofs of these results make use of the theory of linear forms in logarithms ([Bak75], [Dav95]), which is used to show that the periodic points of the maps in question have strong diophantine properties. It is not clear how to apply the theory of linear forms in logarithms in the case of more general rational maps. In this paper, we use Roth's Theorem ([Rot55]) from diophantine approximation in place of the theory of linear forms in logarithms. This allows us to work in greater generality.

0.1. **Statements of the main theorems.** The main results of this paper are generalizations of (0.0.3). Let K be a number field or a function field of characteristic zero, let v be a place of K, and let $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ be a nonconstant rational map of degree $d > 1$. We prove the following equidistribution result for the periodic points of φ.

Theorem 4.6. *For any nonzero polynomial F with coefficients in $\overline{K}$, we have*

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v \, d\mu_{\varphi,v} = \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=[w:1] \\ F(w)\neq 0}} \log|F(w)|_v.$$

This allows us to show that for any point $\beta \in \overline{K}$, the canonical height $h_\varphi(\beta)$ can be computed by taking the average of the log of the absolute value of a minimal polynomial for β over the periodic points of φ.

Theorem 4.9. *For any $\beta \in \overline{K}$ and any nonzero irreducible $F \in K[t]$ such that $F(\beta) = 0$, we have*

$$\begin{aligned}(\deg K)(\deg F)(h_\varphi(\beta) - h_\varphi(\infty)) \\ = \sum_{\textit{places } v \textit{ of } K} \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=[w:1] \\ F(w)\neq 0}} \log|F(w)|_v.\end{aligned}$$

In both the theorems, the w are counted with multiplicity. We explain what multiplicity means in this context in Section 1.

We are also able to prove that $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v \, d\mu_{\varphi,v}$ is the limit as n goes to infinity of the average of $\log|F|_v$ on the points w for which

$\varphi^n(w) = \alpha$, where α is an algebraic point that is not an exceptional point for φ. We state this in Theorem 4.5. This enables us to prove Theorem 4.8, which is the analog of Theorem 4.9 for the points w such that $\varphi^n(w) = \alpha$.

0.2. **Outline of the paper.** This paper is organized as follows:
1 - Notation and terminology.
2 - Brolin-Lyubich integrals and local heights.
3 - Preliminaries from diophantine approximation.
4 - Main results: 4.1 - Using Roth's Theorem; 4.2 - Preperiodic points; 4.3 - Proofs of the main theorems.
5 - A counterexample.
6 - Applications: 6.1 - Lyapunov exponents; 6.2 - Symmetry of canonical heights; 6.3 - Computing with points of small height.

The strategy of the proof of the main theorems is fairly simple. By additivity, it suffices to prove our results for polynomials of the form $F(t) = t - \beta$ for $\beta \in \overline{K}$. After Section 2, we are reduced to showing that

(0.0.4)
$$\lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=[w:1] \\ w\neq\beta}} \log|w-\beta|_v = \lim_{k\to\infty} \frac{\log\max(|P_k(\beta,1)|_v, |Q_k(\beta,1)|_v)}{d^k} - \lim_{k\to\infty} \frac{\log\max(|P_k(1,0)|_v, |Q_k(1,0)|_v)}{d^k},$$

where φ^k is written as

$$\varphi^k([T_0 : T_1]) = [P_k(T_0,T_1) : Q_k(T_0,T_1)]$$

for homogeneous polynomials P_n and Q_n in the $K[T_0,T_1]$. The points w for which $\varphi^k(w) = w$ are just the solutions to the equation $P_k(w,1) - wQ_k(w,1) = 0$. Thus, we get the left-hand side of (0.0.4) by taking the limit of $\log|P_k(\beta,1) - \beta Q_k(\beta,1)|_v/d^k$ as k goes to ∞. For each k, we rewrite this as

$$\frac{\log|Q_k(\beta,1)|_v}{d^k} + \frac{\log|\frac{P_k(\beta,1)}{Q_k(\beta,1)} - \beta|_v}{d^k}$$

and use diophantine approximation to show that the second term in the equation above usually goes to 0 as $k \to \infty$; our theorems then follow after a bit of calculation. The diophantine approximation result we use is Roth's Theorem, which we state in Section 3 as Theorem 3.1. We use Roth's Theorem to derive Lemma 4.1, which is the key lemma in our proofs of the main theorems. The idea for the proof of Lemma 4.1

comes from Siegel's famous paper [Sie29]. Propositions 4.3 and 4.4 deal with the additional complications that may arise when the β in (0.0.4) is preperiodic. These complications are overcome with somewhat lengthy – but essentially basic – calculations that are very similar to some of the computations carried out by Morton and Silverman in [MS95].

In Section 5, we construct a simple counterexample that shows that Theorem 4.6 will not hold in general when the polynomial F does not have algebraic coefficients. We construct a transcendental number β such that the limit $\lim_{k\to\infty} \frac{1}{2^k} \sum_{\xi^{2^k}=1} \log|\xi - \beta|$ does not exist. This means that there is no way to prove the main results of this paper without using some special properties of algebraic numbers.

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1. Notation and terminology

We fix the following notation:

- K is a number field or a function field of characteristic 0 (by function field we mean a finite algebraic extension of a field of the form $K_{\text{cons}}(T)$ where K_{cons} is algebraically closed in K);
- v is a place of K;
- K_v is the completion of K at v;
- $\mathbb{C}_v$ is the completion of an algebraic closure of K_v at v;
- $\overline{K}$ is the algebraic closure of K in $\mathbb{C}_v$ (note that this means that v extends to all of $\overline{K}$);
- $n_v = [K_v : \mathbb{Q}_v]$ if K is a number field;
- $n_v = 1$ if K is a function field;
- $\deg K = [K : \mathbb{Q}]$ if K is a number field;
- $\deg K = 1$ if K is a function field.

We let $|\cdot|_v$ be an absolute value on $\mathbb{C}_v$ corresponding to v. When K is a function field and π_v generates the maximal prime $\mathcal{M}_v$ in the local ring $\mathcal{O}_v$ corresponding to v, we specify that

$$|\pi_v|_v = e^{-[(\mathcal{O}_v/\mathcal{M}_v):K_{\text{cons}}]},$$

where K_{cons} is the field of constants in K. When K is a number field and v is nonarchimedean, we normalize $|\cdot|_v$ so that

$$|p|_v = p^{-n_v}$$

when v lies over p. When K is a number field and v is archimedean we normalize so that $|\cdot|_v = |\cdot|^{n_v}$ on $\mathbb{Q}$, where $|\cdot|$ is the usual archimedean absolute value on $\mathbb{Q}$.

Throughout this paper, we will work with a nonconstant morphism $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ of degree $d > 1$. We choose homogeneous polynomials $P, Q \in K[T_0, T_1]$ of degree d and a coordinate system $[s : t]$ for $\mathbb{P}^1_{\overline{K}}$ such that

$$\varphi([T_0 : T_1]) = [P(T_0, T_1) : Q(T_0, T_1)],$$

where P and Q have no common zero in $\mathbb{P}^1(\overline{K})$. We let $P_1 = P$ and $Q_1 = Q$, and for $k \geq 2$ we define P_k and Q_k recursively by

$$P_k(T_0, T_1) = P_{k-1}(P(T_0, T_1), Q(T_0, T_1))$$

and

$$Q_k(T_0, T_1) = Q_{k-1}(P(T_0, T_1), Q(T_0, T_1)).$$

Having chosen coordinates, we can define the usual Weil height as

$$h([a : b]) = \frac{1}{\deg K} \sum_{\text{places } v \text{ of } K} \log \max(|a|_v, |b|_v)$$

when $a, b \in K$. When a and b lie in an extension L of K, this definition extends to

(1.0.5)

$$h([a : b]) = \frac{1}{[L : K](\deg K)} \sum_{\text{places } w \text{ of } L} [L_w : K_v] \log \max(|a|_w, |b|_w),$$

where L_w is the completion of L at w and the absolute value $|\cdot|_w$ restricts to some $|\cdot|_v$ on K.

As in [CS93], we define the canonical height h_φ as

$$h_\varphi([a : b]) = \lim_{k \to \infty} \frac{h(\varphi^k([a : b]))}{d^k}. \tag{1.0.6}$$

We say that $\alpha \in \mathbb{P}^1(\overline{K})$ is a **periodic** point for φ if there exists a positive integer n such that $\varphi^n(\alpha) = \alpha$. If α is periodic, we define the **period** of α to be the smallest integer ℓ such that $\varphi^\ell(\alpha) = \alpha$. We say that α is **preperiodic** if there exists a positive integer n such that $\varphi^n(\alpha)$ is periodic.

We say that $\alpha \in \mathbb{P}^1(\overline{K})$ is an **exceptional** point for φ if $\varphi^2(\alpha) = \alpha$ and φ^2 is totally ramified at α. This is equivalent to saying that the set $\bigcup_{k=1}^\infty (\varphi^k)^{-1}(\alpha)$ is finite. If α is exceptional, then at each place v, there is a v-adically open set $\mathcal{U}$ containing α such that the sequence $(\varphi^{\ell k}(\beta))_k$ converges to α for each $\beta \in \mathcal{U}$, where ℓ is the period of α (which is either 1 or 2). We call $\mathcal{U}$ the **attracting basin** of α.

We always count points with multiplicities in this paper. The multiplicity of a point $[z:1]$ in the multi-set $\{w \mid \varphi^k(w) = w\}$ is the highest power of $t-z$ that divides the polynomial $P_k(t,1) - tQ_k(t,1)$. The multiplicity of a point $[z:1]$ in the multi-set $\{w \mid \varphi^k(w) = [s:u]\}$ is the highest power of $t-z$ that divides the polynomial $uP_k(t,1) - sQ_k(t,1)$ (here s, u, and z are taken to be elements of $\overline{K}$, while t is taken to be a variable).

We note that everything done in this paper depends upon our choice of coordinates. In particular, our integrals are closely related to the canonical local heights (see [CG97]) for the point $[1:0]$ at infinitely, so our choice of the point at infinity affects all of our integrals. To emphasize the fact that we treat $[1:0]$ as the point at infinity, we denote it as ∞ where appropriate.

2. Brolin-Lyubich integrals and local heights

As noted in the introduction, Brolin [Bro65] and Lyubich [Lyu83] have constructed a φ-invariant measure $\mu_{\varphi,v}$ with support on the Julia set of φ, when v is an infinite place. Similarly, Zhang ([Zha95]) has constructed a distribution $\mu_{\varphi,v}$, which extends to the measure of Brolin and Lyubich, as is shown in [PST04, Section 7]. Zhang also shows that that for any polynomial F, the integral $\int_{\mathbb{P}^1(\mathbb{C})} \log|F|_v \, d\mu_{\varphi,v}$ exists. With this notation, we have the following.

Proposition 2.1. *Let v be an infinite place of a number field K and let $F(t) = t - \beta$ for $\beta \in \mathbb{C}_v$. Then*

$$\begin{aligned}\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{\varphi,v} = &\lim_{k\to\infty} \frac{\log\max(|P_k(\beta,1)|_v, |Q_k(\beta,1)|_v)}{d^k} \\ &- \lim_{k\to\infty} \frac{\log\max(|P_k(1,0)|_v, |Q_k(1,0)|_v)}{d^k}.\end{aligned} \tag{2.1.1}$$

Proof. We use limits of metrics as in [Zha95, Section 2]. Let $\|\cdot\|_{0,v}$ be the metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ such that for any section $\ell = aT_0 + bT_1$ of $\mathcal{O}_{\mathbb{P}^1}(1)$, we have $\|\ell([s:u])\|_{0,v} = \frac{|as+bu|_v}{\max(|s|_v,|u|_v)}$. Let f be the isomorphism between $\varphi^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^1}(d)$ given by $f : \varphi^*T_0 \mapsto P(T_0,T_1)$ and $f : \varphi^*T_1 \mapsto Q(T_0,T_1)$. Then, if we define $\|\cdot\|_{n,v} = f\varphi^*\|\cdot\|_{n-1,v}^{1/d}$ for $n \geq 1$, the metrics $\|\cdot\|_{n,v}$ converge to a metric $\|\cdot\|_{\varphi,v}$, which does not depend on our original choice of $\|\cdot\|_{0,v}$, by [Zha95, Theorem 2.2]. It is easy to check that for any global section $\ell = aT_0 + bT_1$ of $\mathcal{O}_{\mathbb{P}^1}(1)$, we have

$$\|\ell([s:u])\|_{\varphi,v} = \lim_{k\to\infty} \frac{|as+bu|_v}{\max\left(|P_k(s,u)|_v, |Q_k(s,u)|_v\right)^{1/d^k}}. \tag{2.1.2}$$

Now pick a section $\ell = T_0 + bT_1$ that does not vanish at $[1 : 0]$ or $[\beta : 1]$. Note that $\|\cdot\|_{\varphi,v}$ is also a limit of smooth metrics; for example we can define $\|\cdot\|'_{0,v}$ to be the Fubini-Study metric, and let $\|\cdot\|'_{n,v} = f\varphi^*\|\cdot\|'^{1/d}_{n-1,v}$. Then we have $d\mu_{\varphi,v} = \lim_{k\to\infty} \frac{-d\bar{d}\log\|\cdot\|'_{k,v}}{2\pi i}$, so we have

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{\varphi,v} = \log|F(-b)|_v - \log\|\ell(\beta,1)\|_{\varphi,v} + \log\|\ell(1,0)\|_{\varphi,v},$$

by [PST04, Proposition B.2] (see also [Lan88, Lemma 2.1.1, pp. 22-23]). Plugging (2.1.2) into this equation gives

(2.1.3)
$$\begin{aligned}\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{\varphi,v} = {} & \log|F(-b)|_v \\ & - \lim_{k\to\infty} \log\left(\frac{|\beta+b|_v}{\max\left(|P_k(\beta,1)|_v, |Q_k(\beta,1)|_v\right)^{1/d^k}}\right) \\ & + \lim_{k\to\infty} \log\left(\frac{|1|_v}{\max\left(|P_k(1,0)|_v, |Q_k(1,0)|_v\right)^{1/d^k}}\right).\end{aligned}$$

We also have $|F(-b)|_v = |-b-\beta|_v$, so $\log|F(-b)|_v - \log|\beta+b|_v = 0$. Thus, (2.1.3) simplifies to (2.1.1), as desired. □

This leads us to the following more general definition for any place v.

Definition 2.2. *Let v be a place of K. Let $F(t) = \gamma\prod_{i=1}^n (t-\beta_i)$, where $\gamma, \beta_1, \ldots, \beta_n \in \mathbb{C}_v$, be a polynomial in $\mathbb{C}_v[t]$. We define*

(2.2.1)
$$\begin{aligned}\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v d\mu_{\varphi,v} = {} & \sum_{i=1}^n \lim_{k\to\infty} \frac{\log\max(|P_k(\beta_i,1)|_v, |Q_k(\beta_i,1)|_v)}{d^k} \\ & - n\lim_{k\to\infty} \frac{\log\max(|P_k(1,0)|_v, |Q_k(1,0)|_v)}{d^k} + \log|\gamma|_v.\end{aligned}$$

This definition is equivalent to Definition 5.2 of [PST04] when φ fixes $[1 : 0]$. Baker and Rumely ([BR05]), Chambert-Loir ([CL04]), and Favre and Rivera-Letelier ([FRL04b] and [FRL04a]) have shown that there is a measure $\mu_{\varphi,v}$ on the Berkovich space (see [Ber90]) corresponding to $\mathbb{P}^1(\mathbb{C}_v)$ that gives rise to these integrals. Note that although our integrals are defined for points in $\mathbb{C}_v$, the results we prove in Section 4 apply only to points in $\overline{K}$.

Call and Goldstine ([CG97, Theorem 3.1]) have shown that

$$\hat{h}_{\varphi,v}([\beta:1]) = \lim_{k\to\infty} \frac{\log\max(|P_k(\beta,1)|_v, |Q_k(\beta,1)|_v)}{d^k}$$

is the unique Weil function for $[1:0]$ at v (see [Lan83, Chpater 10] for a definition of Weil functions) that satisfies

$$\hat{h}_{\varphi,v}(\varphi([a:b])) = d\hat{h}_{\varphi,v}([a:b]) + \log\left|\frac{Q(a,b)}{b^n}\right|_v,$$

for any $b \neq 0$ (see [CG97, Theorem 2.1]). The function $\hat{h}_{\varphi,v}(\cdot)$ is called a canonical local height for φ.

3. Preliminaries from diophantine approximation

The following well-known theorem of Roth is the principal tool from diophantine approximation that is used in this paper.

Theorem 3.1. *[*[Rot55]*] If $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$, then for any $\epsilon > 0$, there is a constant C such that*

$$\left|\alpha - \frac{a}{b}\right| > \frac{C}{|b|^{2+\epsilon}},$$

for all $a/b \in \mathbb{Q}$ such that $a/b \neq \alpha$.

We will need to work in slightly generality. In the terminology of the previous section, Roth's admits the following generalization (see [Lan83, Theorem 7.1.1]), which holds when K is number field or a function fields of characteristic 0.

Theorem 3.2. *Let $\alpha_1, \ldots, \alpha_n$ be elements of $\overline{K}$ and let $L \subset \overline{K}$ be a finite extension of K. Then, for any $\epsilon > 0$, we have*

$$\frac{1}{[L:K](\deg K)} \sum_{i=1}^{n} \max(0, -\log|\alpha_i - \beta|_v^{[L_w:K_v]n_v}) \leq (2+\epsilon)h(\beta) + O(1),$$

for all $\beta \in L$ not in the set $\{\alpha_1, \ldots, \alpha_n\}$.

We will work with divisors on $\mathbb{P}^1_{\overline{K}}$ rather than elements of $\overline{K}$. For a divisor $D = \sum_{i=1}^n m_i\alpha_i$, where $\alpha_i \in \mathbb{P}^1(\overline{K})$, we define

$$r(D) = \max_i(m_i).$$

Lemma 3.3. *Let D be a divisor such that* $\operatorname{Supp} D$ *does not contain any exceptional points of φ. Then $\lim_{k\to\infty} \frac{r((\varphi^k)^* D)}{d^k} = 0$.*

Proof. Recall that α is an exceptional point if and only if $\varphi^2(\alpha) = \alpha$ and φ is totally ramified at both α and $\varphi(\alpha)$. Since φ has at most two totally ramified points, it follows that if α is not exceptional, then one of α, $\varphi(\alpha)$, and $\varphi^2(\alpha)$ is not a totally ramified point of φ. Thus, for any divisor E such that $\operatorname{Supp} E$ does not contain an exceptional point, $r((\varphi^3)^*E) \leq d^2(d-1)r(E)$. Now, since $\operatorname{Supp} D$ does not contain an exceptional point, $\operatorname{Supp}(\varphi^k)^*D$ does not contain an exceptional point for any k. Thus, for any $k \geq 0$, we see that $\frac{r((\varphi^k)^*D)}{d^k}$ is less than or equal to $((d-1)/d)^{(k-2)/3} r(D)$, which goes to zero as k goes to infinity. □

Let $[a:1]$ be a point in $\mathbb{P}^1(\overline{K})$. Then for any $[b:1] \neq [a:1]$ in $\mathbb{P}^1(\mathbb{C}_v)$, we let

$$\lambda_{[a:1],v}([b:1]) = \max(-\log|b-a|_v, 0).$$

We extend this definition to the point at $[1:0]$ by letting

$$\lambda_{[a:1],v}([1:0]) = 0.$$

and

$$\lambda_{[1:0],v}([b:1]) = \max(0, \log|b|_v).$$

Let $D = \sum_{i=1}^n m_i \alpha_i$, where $\alpha_i \in \mathbb{P}^1(\overline{K})$ and $m_i \in \mathbb{Z}$. We let

$$\lambda_{D,v}(\beta) = \sum m_i \lambda_{\alpha_i,v}(\beta)$$

for points $\beta \in \mathbb{P}^1(\mathbb{C}_v)$ that are not in $\operatorname{Supp} D$. Then $\lambda_{D,v}$ is a **Weil function** for D at v as defined in [Lan83, Chapter 10]. It is easy to check that for any divisor D and any rational map φ on $\mathbb{P}^1$, we have

$$\lambda_{D,v}(\varphi(\beta)) \leq \lambda_{\varphi^*D,v}(\beta) + O(1), \tag{3.3.1}$$

for all $\beta \in \mathbb{P}^1(\overline{K})$ away from the support of D and φ^*D. This is a general functorial property of Weil functions, as explained in [Lan83, Chapter 10].

With this terminology, it follows from Theorem 3.2 that for any $\epsilon > 0$, any finite extension L of K, and any positive divisor D on $\mathbb{P}^1(\overline{K})$ with $r(D) = 1$, we have

$$\frac{1}{[L:K](\deg K)} \lambda_{D,v}(\beta) \leq (2+\epsilon)h(\beta) + O(1)$$

for all $\beta \in \mathbb{P}^1(L)$ away from the support of D. Hence, for any positive divisor D we have

$$\frac{1}{[L:K](\deg K)} \lambda_{D,v}(\beta) \leq r(D)(2+\epsilon)h(\beta) + O(1). \tag{3.3.2}$$

4. Main results

4.1. **Using Roth's Theorem.** Roth's Theorem allows us to prove the following lemma. The idea of the proof is that if $\varphi^{k+\ell}(\beta)$ approximates D very closely, then $\varphi^k(\beta)$ approximates $(\varphi^\ell)^*D$ very closely. Since $\varphi^k(\beta)$ has height approximately equal to $1/d^\ell$ times the height of $\varphi^{k+\ell}(\beta)$, this makes $h(\varphi^k(\beta))$ small relative to $\lambda_{(\varphi^\ell)^*D}(\beta)$. Repeating this for infinitely many $\varphi^k(\beta)$ gives a contradiction to Roth's Theorem. This idea is due to Siegel ([Sie29]).

Lemma 4.1. *Let D be a positive divisor on $\mathbb{P}^1$ such that* $\operatorname{Supp} D$ *does not contain any of the exceptional points of φ. Let β be a point in $\mathbb{P}^1(\overline{K})$ for which there is a strictly increasing sequence of integers $(e_i)_{i=1}^\infty$ such that $\varphi^{e_i}(\beta) \notin \operatorname{Supp} D$. Then*

$$\lim_{i\to\infty} \frac{\lambda_{D,v}(\varphi^{e_i}(\beta))}{d^{e_i}} = 0. \tag{4.1.1}$$

Proof. Let L be a finite extension of K for which $\beta \in \mathbb{P}^1(L)$. Choose $\delta > 0$. By Lemma 3.3, we may pick an integer ℓ such that $\frac{r((\varphi^\ell)^*D)}{d^\ell} < \delta/2$. We may then write $\frac{r((\varphi^\ell)^*D)(2+\epsilon)}{d^\ell} = \delta$ for some $\epsilon > 0$. For any e_i, we have $\varphi^{e_i-\ell}(\beta) \notin \operatorname{Supp}(\varphi^\ell)^*D$ since $\varphi^{e_i}(\beta) \notin \operatorname{Supp} D$. Thus, applying Roth's Theorem (as expressed in (3.3.2)), we find that for all e_i we have

$$\frac{1}{[L:K](\deg K)}\lambda_{(\varphi^\ell)^*D,v}(\varphi^{e_i-\ell}(\beta)) \le r((\varphi^\ell)^*D)(2+\epsilon)h(\varphi^{e_i-\ell}(\beta))+O(1).$$

Using (3.3.1) and the fact that $h(\varphi^{e_i}(\beta)) \le d^\ell h_\varphi(\varphi^{e_i-\ell}(\beta)) + O(1)$, we then obtain

$$\begin{aligned}
\frac{1}{[L:K](\deg K)}\lambda_{D,v}(\varphi^{e_i}(\beta)) &\le \frac{1}{[L:K](\deg K)}\lambda_{(\varphi^\ell)^*D,v}(\varphi^{e_i-\ell}(\beta)) + O(1)\\
&\le r((\varphi^\ell)^*D)(2+\epsilon)h(\varphi^{e_i-\ell}(\beta)) + O(1)\\
&\le \frac{r((\varphi^\ell)^*D)(2+\epsilon)}{d^\ell}h(\varphi^{e_i}(\beta)) + O(1)\\
&\le \delta h(\varphi^{e_i}(\beta)) + O(1)\\
&\le \delta d^{e_i} h(\beta) + O(1).
\end{aligned}$$

Dividing through by d^{e_i} gives

$$\limsup_{i\to\infty} \frac{\lambda_{D,v}(\varphi^{e_i}(\beta))}{d^{e_i}} \le [L:K](\deg K)\delta h(\beta).$$

Since $\lambda_{D,v}(\varphi^{e_i}(\beta)) \ge 0$, letting δ go to zero gives (4.1.1), as desired. □

This allows us to prove the following Proposition, which will be used to prove Theorems 4.5 and 4.6.

Proposition 4.2. *Let $\alpha = [s : u]$ be a nonexceptional point in $\mathbb{P}^1(\overline{K})$. Let $\beta = [a : b]$ be a point in $\mathbb{P}^1(\overline{K})$ for which there is a strictly increasing sequence of integers $(e_i)_{i=1}^\infty$ such that $\varphi^{e_i}(\beta) \neq \alpha$. Then*

$$\lim_{i\to\infty} \frac{\log |uP_{e_i}(a,b) - sQ_{e_i}(a,b)|_v}{d^{e_i}} = \lim_{i\to\infty} \frac{\log \max(|P_{e_i}(a,b)|_v, |Q_{e_i}(a,b)|_v)}{d^{e_i}}$$

Proof. If $[1 : 0]$ is an exceptional point of φ, let $\mathcal{U}$ be its attracting basin; if $[1 : 0]$ is not exceptional let $\mathcal{U}$ simply equal $\{[1 : 0]\}$.

Let $(\ell_j)_{j=1}^\infty$ be the subsequence of $(e_i)_{i=1}^\infty$ such that $\varphi^{\ell_j}(\beta) \notin \mathcal{U}$ for all j (this subsequence may be empty). We have

$$\lim_{j\to\infty} \frac{\log |P_{\ell_j}(a,b)/Q_{\ell_j}(a,b)|_v}{d^{\ell_j}} = 0. \tag{4.2.1}$$

If $[1 : 0]$ is not exceptional, this follows from Lemma 4.1 with $D = [1 : 0]$. If $[1 : 0]$ is exceptional, the fact that $\varphi^{\ell_j}(\beta) \notin \mathcal{U}$ for all j implies that $|P_{\ell_j}(a,b)/Q_{\ell_j}(a,b)|_v$ is bounded for all j, so (4.2.1) clearly holds. It follows immediately from (4.2.1) that

$$\lim_{j\to\infty} \frac{\log \max(|P_{\ell_j}(a,b)|_v, |Q_{\ell_j}(a,b)|_v)}{d^{\ell_j}} = \lim_{j\to\infty} \frac{\log |Q_{\ell_j}(a,b)|_v}{d^{\ell_j}}. \tag{4.2.2}$$

Note that if $u = 0$, then

$$uP_{\ell_j}(a,b) - sQ_{\ell_j}(a,b) = sQ_{\ell_j}(a,b),$$

so we are done. Otherwise, by Lemma 4.1, we have

$$\lim_{j\to\infty} \frac{\max\left(0, -\log\left|\frac{P_{\ell_j}(a,b)}{Q_{\ell_j}(a,b)} - \frac{s}{u}\right|_v\right)}{d^{\ell_j}} = 0.$$

Combining this with (4.2.1), we see that

$$\lim_{j\to\infty} \frac{\log\left|\frac{P_{\ell_j}(a,b)}{Q_{\ell_j}(a,b)} - \frac{s}{u}\right|_v}{d^{\ell_j}} = 0.$$

Thus, using (4.2.2), we obtain

$$\begin{aligned}
&\lim_{j\to\infty} \frac{\log|uP_{\ell_j}(a,b) - sQ_{\ell_j}(a,b)|_v}{d^{\ell_j}} \\
&= \lim_{j\to\infty} \frac{\log\left(|Q_{\ell_j}(a,b)|_v |u|_v \left|\frac{P_{\ell_j}(a,b)}{Q_{\ell_j}(a,b)} - \frac{s}{u}\right|_v\right)}{d^{\ell_j}} \\
&= \lim_{j\to\infty} \frac{\log|Q_{\ell_j}(a,b)|_v}{d^{\ell_j}} + \lim_{j\to\infty} \frac{\log\left|\frac{P_{\ell_j}(a,b)}{Q_{\ell_j}(a,b)} - \frac{s}{u}\right|_v}{d^{\ell_j}} \\
&= \lim_{j\to\infty} \frac{\log\max(|P_{\ell_j}(a,b)|_v, |Q_{\ell_j}(a,b)|_v)}{d^{\ell_j}},
\end{aligned}$$

as desired.

Now, let $(m_j)_{j=1}^{\infty}$ be be the subsequence of $(e_i)_{i=1}^{\infty}$ consisting of all integers such that $\varphi^{m_j}(\beta) \in \mathcal{U}$ (this subsequence may be empty). If $\alpha = [1:0]$, then $[1:0]$ is not exceptional by assumption, so there are no m_j and we are done. Otherwise, we have

$$\lim_{j\to\infty} \frac{|sQ_{m_j}(a,b)|_v}{|uP_{m_j}(a,b)|_v} = 0,$$

which means that

$$\begin{aligned}
&\lim_{j\to\infty} \frac{\log|uP_{m_j}(a,b) - sQ_{m_j}(a,b)|_v}{d^{m_j}} \\
&= \lim_{j\to\infty} \frac{\log|uP_{m_j}(a,b)|_v}{d^{m_j}} \\
&= \lim_{j\to\infty} \frac{\log\max(|P_{m_j}(a,b)|_v, |Q_{m_j}(a,b)|_v))}{d^{m_j}}.
\end{aligned}$$

Since every element of the sequence $(e_i)_{i=1}^{\infty}$ is in $(\ell_j)_{j=1}^{\infty}$ or $(m_j)_{j=1}^{\infty}$, this completes our proof. □

4.2. **Preperiodic points.** Proposition 4.2 provides all the information we need when $\varphi^k([a:b]) = [s:u]$ for at most finitely many k. When $[s:u]$ is preperiodic, however, there may be infinitely many k such that $\varphi^k([a:b]) = [s:u]$. New complications arise when this is the case; we treat these complications in Propositions 4.3 and 4.4.

Suppose that $(bT_0 - aT_1)^{w_k}$ is the highest power of $(bT_0 - aT_1)$ that divides $uP_k(T_0,T_1) - sQ_k(T_0,T_1)$ in $\overline{K}[T_0,T_1]$. We write

$$uP_k(T_0,T_1) - sQ_k(T_0,T_1) = (bT_0 - aT_1)^{w_k} G_k(T_0,T_1)$$

where G_k is a polynomial in $\overline{K}[T_0,T_1]$ such that $G_k(a,b) \neq 0$.

Proposition 4.3. *Let $[s:u]$ be a nonexceptional point of φ. Then, with notation as above, we have*

$$\lim_{k\to\infty} \frac{\log |G_k(a,b)|_v}{d^k} = \lim_{k\to\infty} \frac{\log \max(|P_k(a,b)|_v, |Q_k(a,b)|_v)}{d^k}. \tag{4.3.1}$$

Proof. By Proposition 4.2, equation (4.3.1) holds if we restrict to the k for which $\varphi^k([a:b]) \neq \alpha$. If there are at most finitely many k such that $\varphi^k([a:b]) = \alpha$, we are therefore done. Otherwise, let j be the smallest positive integer such that $\varphi^j([\beta:1]) = \alpha$ and let ℓ be the period of α. Then $\varphi^k([\beta:1]) = \alpha$ precisely when k is of the form $j + m\ell$ for some integer $m \geq 0$. If $\varphi^\ell([s:u]) = [s:u]$, then $uT_0 - sT_1$ divides $uP_\ell(T_0,T_1) - sQ_\ell(T_0,T_1)$.

Suppose that $u \neq 0$. Then, expanding Q_ℓ out in the variables $uT_0 - sT_1$ and T_1, we see that since $uT_0 - sT_1$ cannot divide $Q_\ell(T_0,T_1)$ (because if it did, then it would also divide $P_\ell(T_0,T_1)$ and we know that Q_ℓ and P_ℓ have no factors). Hence, we have

$$Q_\ell(T_0,T_1) = g_0 T_1^{d^\ell} + (uT_0 - sT_1)W(T_0,T_1)$$

for some nonzero $g_0 \in \overline{K}$ and some $W(T_0,T_1) \in \overline{K}[T_0,T_1]$. For any $m \geq 0$ we thus have

$$Q_{m\ell} = g_0 (Q_{(m-1)\ell})^{d^\ell} + (uP_{(m-1)\ell} - sQ_{(m-1)\ell}) W(P_{(m-1)\ell}, Q_{(m-1)\ell}).$$

Using induction, we see then that

$$Q_{m\ell}(T_0,T_1) = g_0^{\sum_{i=0}^{m-1} d^{i\ell}} T_1^{d^{m\ell}} + (uT_0 - sT_1) W_m(T_0,T_1), \tag{4.3.2}$$

for some polynomial $W_m(T_0,T_1) \in \overline{K}[T_0,T_1]$. Similarly, we may write

$$\begin{aligned} &uP_\ell(T_0,T_1) - sQ_\ell(T_0,T_1) \\ &= (uT_0 - sT_1)^r f_r T_1^{d-r} + (uT_0 - sT_1)^{r+1} V(T_0,T_1), \end{aligned} \tag{4.3.3}$$

for some nonzero $f_r \in \overline{K}$, some integer $r > 0$, and some $V(T_0,T_1)$ in $\overline{K}[T_0,T_1]$. Then for any m, we have

$$\begin{aligned} uP_{m\ell} - sQ_{m\ell} &= (uP_{(m-1)\ell} - sQ_{(m-1)\ell})^r f_r Q_{(m-1)\ell}^{d-r} \\ &\quad + (P_{(m-1)\ell} - sQ_{(m-1)\ell})^{r+1} V(P_{(m-1)\ell}, Q_{(m-1)\ell}), \end{aligned}$$

so, using (4.3.2), (4.3.3), and induction, we obtain

$$\begin{aligned} &uP_{m\ell}(T_0,T_1) - sQ_{m\ell}(T_0,T_1) \\ &= (uT_0 - sT_1)^{r^m} f_r^{\sum_{i=0}^{m-1} r^i} T_1^{d^{m\ell} - r^m} g_0^{\sum_{i=0}^{m-1}(d^{i\ell} - r^i)} \\ &+ (uT_0 - sT_1)^{r^m+1} Z_m(T_0,T_1), \end{aligned} \tag{4.3.4}$$

for Z_m a polynomial in $\overline{K}[T_0, T_1]$. Since $r < d^\ell$, we have

$$\lim_{m\to\infty} \frac{\log |f_r^{\sum_{i=0}^{m-1} r^i} g_0^{\sum_{i=0}^{m-1}(d^{i\ell}-r^i)}|_v}{d^{m\ell}} = \lim_{m\to\infty} \frac{\log |g_0^{\sum_{i=0}^{m-1} d^{i\ell}}|_v}{d^{m\ell}} = \frac{\log |g_0|_v}{d^\ell - 1}.$$

Now, let ϵ be the highest power of $aT_0 - bT_1$ that divides $uP_j - sQ_j$. Using (4.3.4), we see that we have

$$uP_{j+m\ell}(T_0, T_1) - sQ_{j+m\ell}(T_0, T_1) = (bT_0 - aT_1)^{\epsilon r^m} G_{j+m\ell}(T_0, T_1)$$

for a polynomial $G_{j+m\ell} \in \overline{K}[T_0, T_1]$. Letting m go to infinity, we see from (4.3.4) that

$$\lim_{m\to\infty} \frac{\log |G_{j+m\ell}(a,b)|_v}{d^{j+m\ell}} = \frac{\log |g_0|_v}{d^j(d^\ell - 1)} + \frac{\log |Q_j(a,b)|_v}{d^j}.$$

Similarly, (4.3.2) yields

$$\lim_{m\to\infty} \frac{\log |Q_{j+m\ell}(a,b)|_v}{d^{j+m\ell}} = \frac{\log |g_0|_v}{d^j(d^\ell - 1)} + \frac{\log |Q_j(a,b)|_v}{d^j}.$$

Moreover, since $uP_{j+m\ell}(a,b) = sQ_{j+m\ell}(a,b)$ for every m, we have

$$\lim_{m\to\infty} \frac{\log |P_{j+m\ell}(a,b)|_v}{d^{j+m\ell}} = \lim_{m\to\infty} \frac{\log |Q_{j+m\ell}(a,b)|_v}{d^{j+m\ell}}.$$

Hence

$$\lim_{m\to\infty} \frac{\log |G_{j+m\ell}(a,b)|_v}{d^{j+m\ell}} = \lim_{m\to\infty} \frac{\log \max(|P_{j+m\ell}(a,b)|_v, |Q_{j+m\ell}(a,b)|_v)}{d^{j+m\ell}},$$

which completes our proof in the case $u \neq 0$. The proof in the case $u = 0$ proceeds in exactly the same way, using T_0 in place of T_1. □

We have a similar result for the polynomials $T_0P_k - T_1Q_k$. We write

$$T_0P_k(T_0, T_1) - T_1Q_k(T_0, T_1) = (bT_0 - aT_1)^{n_k} H_k(T_0, T_1)$$

where H_k is a polynomial in $\overline{K}[T_0, T_1]$ such that $H_k(a,b) \neq 0$. The proof of the following proposition is similar to Morton's and Silverman's proof of [MS95, Lemma 3.4], but it requires a bit more detail since it yields information about $H_k(a,b)$ as well as n_k.

Proposition 4.4. *With notation as above, we have*

$$\lim_{k\to\infty} \frac{\log |H_k(a,b)|_v}{d^k} = \lim_{k\to\infty} \frac{\log \max(|P_k(a,b)|_v, |Q_k(a,b)|_v)}{d^k}. \tag{4.4.1}$$

Furthermore, n_k remains bounded as k goes to infinity.

Proof. If $(e_i)_{i=1}^{\infty}$ is a strictly increasing sequence of integers such that $\varphi^{e_i}([a:b]) \neq [a:b]$ for each e_i, then

$$H_{e_i}(T_0, T_1) = T_0 P_{e_i}(T_0, T_1) - T_1 Q_{e_i}(T_0, T_1)$$

for all e_i. Hence, by Proposition 4.2, we

$$\lim_{i\to\infty} \frac{\log |H_{e_i}(a,b)|_v}{d^{e_i}} = \lim_{i\to\infty} \frac{\log \max(|P_{e_i}(a,b)|_v, |Q_{e_i}(a,b)|_v)}{d^{e_i}}.$$

If $[a:b]$ is not periodic, this finishes the proof. Thus, we may assume that $[a:b]$ is periodic. The rest of the proof is a computation. We divide it into three steps.

Step I. We begin by changing variables so that $[a:b]$ becomes $[0:1]$. If $b = 0$, we write $U_0 = T_1/a$ and $U_1 = -T_0$. We then let

$$R(U_0, U_1) = \frac{1}{a} Q(T_0, T_1)$$

and

$$S(U_0, U_1) = -P(T_0, T_1)$$

(this is simply the inverse of the transformation we defined on T_0 and T_1 – our change of variables is obtained by conjugation by a change-of-basis matrix). If $b \neq 0$, we write $U_1 = \frac{1}{b} T_1$ and

$$U_0 = bT_0 - aT_1.$$

We then let $S(U_0, U_1) = Q(T_0, T_1)/b$ and

$$R(U_0, U_1) = bP(T_0, T_1) - aQ(T_0, T_1).$$

We define R_k and S_k recursively by letting $R_1 = R$, $S_1 = S$, and setting

$$R_{k+1}(U_0, U_1) = R_k(R(U_0, U_1), S(U_0, U_1))$$

and

$$S_{k+1}(U_0, U_1) = S_k(R(U_0, U_1), S(U_0, U_1)).$$

By the construction of our change of variables, we have

$$U_1 R_k(U_0, U_1) - U_0 S_k(U_0, U_1) = T_0 P_k(T_0, T_1) - T_1 Q_k(T_0, T_1) \tag{4.4.2}$$

as polynomials in T_0 and T_1. Hence, if $U_0^{n_k}$ is the highest power of U_0 that divides $U_1 R_k(U_0, U_1) - U_0 S_k(U_0, U_1)$ and τ_k is the coefficient of the $U_0^{n_k} U_1^{d^k - n_k}$ term in $U_1 R_k(U_0, U_1) - U_0 S_k(U_0, U_1)$, then

$$\tau_k = H_k(a, b).$$

Now, let ℓ be the smallest positive integer for which $\varphi^\ell([a:b]) = [a:b]$. Note that $|S_{m\ell}(1,0)|_v = \frac{|Q_{m\ell}(a,b)|_v}{|b|_v}$ if $b \neq 0$ and $|S_{m\ell}(1,0)|_v = |P_{m\ell}(a,b)|_v/|a|_v$ otherwise. Since

$$[P_{m\ell}(a,b) : Q_{m\ell}(a,b)] = [a:b]$$

for every m, it follows that

$$\lim_{m\to\infty} \frac{\log |S_{m\ell}(0,1)|_v}{d^{m\ell}} = \lim_{m\to\infty} \frac{\log \max(|P_{m\ell}(a,b)|_v, |Q_{m\ell}(a,b)|_v)}{d^{m\ell}}.$$

Thus, it will suffice to show that

$$\lim_{m\to\infty} \frac{\log |\tau_{m\ell}|_v}{d^{m\ell}} = \lim_{m\to\infty} \frac{\log |S_{m\ell}(0,1)|_v}{d^{m\ell}}. \tag{4.4.3}$$

We write

$$R_\ell(U_0,U_1) = \sum_{i=1}^{d^\ell} f_i U_0^i U_1^{d^\ell - i}$$

(note that U_0 divides R_ℓ by our change of variables) and

$$S_\ell(U_0,U_1) = \sum_{i=0}^{d^\ell} g_i U_0^i U_1^{d^\ell - i}.$$

Using induction, we see that

$$R_{m\ell}(U_0,U_1) \equiv f_1^m g_0^{(\sum_{j=0}^{m-1} d^{j\ell})-m} U_0 U_1^{d^{m\ell}-1} \pmod{U_0^2}$$

and

$$S_{m\ell}(U_0,U_1) \equiv g_0^{\sum_{j=0}^{m-1} d^{j\ell}} U_1^{d^{m\ell}} \pmod{U_0^2}.$$

Thus, we have

$$\begin{aligned} &U_1 R_{m\ell}(U_0,U_1) - U_0 S_{m\ell}(U_0,U_1) \\ &\equiv g_0^{\sum_{j=0}^{m-1} d^{j\ell}} ((f_1/g_0)^m - 1) U_0 U_1^{d^{m\ell}} \pmod{U_0^2}. \end{aligned} \tag{4.4.4}$$

Step II. We will now treat the m for which $(f_1/g_0)^m \neq 1$ We have

$$|\log |(f_1/g_0)^m - 1|_v \leq h((f_1/g_0)^m - 1) \leq 2m[K(f_1/g_0):K]h(f_1/g_0)$$

for all m such that $(f_1/g_0)^m \neq 1$ (this is a simple version of Liouville's theorem), so

$$\lim_{\substack{m\to\infty \\ (f_1/g_0)^m \neq 1}} \frac{\log |(f_1/g_0)^m - 1|_v}{d^{m\ell}} = 0.$$

Thus, dividing (4.4.4) through by U_0, we obtain

$$\lim_{\substack{m\to\infty \\ (f_1/g_0)^m \neq 1}} \frac{\log |\tau_{m\ell}|_v}{d^{m\ell}} = \lim_{m\to\infty} \frac{\log |g_0^{\sum_{j=0}^{m-1} d^{j\ell}}|_v}{d^{m\ell}} = \lim_{m\to\infty} \frac{\log |S_{m\ell}(0,1)|_v}{d^{m\ell}},$$

as desired.

Step III. We are left with treating the m for which $(f_1/g_0)^m = 1$. Let ρ be the smallest positive integer m such that $(f_1/g_0)^m = 1$ and write $\omega = \rho\ell$. For $q \geq 1$ we write

$$R_{q\omega}(U_0, U_1) = \sum_{i=1}^{d^{q\omega}} x_i^{[q]} U_0^i U_1^{d^{q\omega}-i}$$

(the summation starts at 1 since U_0 divides $R_{q\omega}$) and

$$S_{q\omega}(U_0, U_1) = \sum_{i=0}^{d^{q\omega}} y_i^{[q]} U_0^i U_1^{d^{q\omega}-i}.$$

Since $f_1^\rho = g_0^\rho$ by assumption, we have $y_0^{[1]} = x_1^{[1]}$ by (4.4.4). Multiplying R_ω and S_ω through by a constant will change all of the limits we are calculating by the same fixed amount, so we may assume that $y_0^{[1]} = x_1^{[1]} = 1$. Let r be the smallest integer greater than 0 such that $x_r^{[1]} \neq y_{r-1}^{[1]}$ (we have $r \geq 2$ since $(f_1/g_0)^m = 1$). Then U_0^r divides $U_1 R_\omega - U_0 S_\omega$, which in turn divides $U_1 R_{q\omega} - U_0 S_{q\omega}$ for any q; hence U_0^r divides $U_1 R_{q\omega} - U_0 S_{q\omega}$ for every q, so $x_j^{[q]} = y_{j-1}^{[q]}$ for $j < r$. To calculate $x_r^{[q]} - y_{r-1}^{[q]}$, we introduce some notation: we let

$$\left(\sum_{i=0}^{M} t_i U_0^i U_1^{M-i} \right)_j = t_j$$

for any polynomial $\sum_{i=0}^{M} t_i U_0^i U_1^{M-i}$. We have

(4.4.5)

$$x_r^{[q]} - y_{r-1}^{[q]}$$
$$= \sum_{i=1}^{r} x_i^{[q-1]} \left((R_\omega)^i (S_\omega)^{d^{(q-1)\omega}-i} \right)_r - \sum_{j=0}^{r-1} y_j^{[q-1]} \left((R_\omega)^j (S_\omega)^{d^{(q-1)\omega}-j} \right)_{r-1}.$$

For any $i < r$, we have $x_i^{[1]} = y_{i-1}^{[1]}$, so $(U_0 R_\omega)_i = (U_1 S_\omega)_i$. Hence, we have

$$\left((R_\omega)^j (S_\omega)^{d^{(q-1)\omega}-j} \right)_{r-1} = \left((R_\omega)^{j+1} (S_\omega)^{d^{(q-1)\omega}-j-1} \right)_r$$

for $j > 0$. For $j = 0$, we have

$$\begin{aligned} \left(S_\omega^{d^{(q-1)\omega}} \right)_{r-1} &= \left(\left(R_\omega + (x_r^{[1]} - y_{r-1}^{[1]}) U_0^r U_1^{d^\omega - r} \right) S_\omega^{d^{(q-1)\omega}-1} \right)_r \\ &= \left(R_\omega S_\omega^{d^{(q-1)\omega}-1} \right)_r + (x_r^{[1]} - y_{r-1}^{[1]}), \end{aligned}$$

since $y_0^{[1]} = x_1^{[1]} = 1$.

Using equation (4.4.5), we see that

$$\begin{aligned} x_r^{[q]} - y_{r-1}^{[q]} &= \sum_{i=1}^{r} x_i^{[q-1]} \left((R_\omega)^i (S_\omega)^{d^{q\omega}-i}\right)_r \\ &- \sum_{j=0}^{r-1} x_{j+1}^{[q-1]} \left((R_\omega)^{j+1} (S_\omega)^{d^{(q-1)\omega}-j-1}\right)_r + (x_r^{[1]} - y_{r-1}^{[1]})(x_1^{[q-1]}) \\ &+ (x_r^{[q-1]} - y_{r-1}^{[q-1]}) \left((R_\omega)^r (S_\omega)^{d^{(q-1)\omega}-r}\right)_r \\ &= (x_r^{[1]} - y_{r-1}^{[1]})(x_1^{[q-1]}) + (x_r^{[q-1]} - y_{r-1}^{[q-1]}), \end{aligned}$$

We have $y_0^{[q-1]} = x_1^{[q-1]} = 1$, since $y_0^{[1]} = x_1^{[1]} = 1$. Thus, assuming inductively that

$$x_r^{[q-1]} - y_{r-1}^{[q-1]} = (q-1)(x_r^{[1]} - y_{r-1}^{[1]}),$$

we have

$$x_r^{[q]} - y_{r-1}^{[q]} = q(x_r^{[1]} - y_{r-1}^{[1]}). \tag{4.4.6}$$

Note in particular that $n_{q\omega} = r$ for all q, so n_k is bounded for all k, as desired.

Now,

$$\lim_{q\to\infty} \frac{\log |q(x_r^{[1]} - y_{r-1}^{[1]})|_v}{d^{q\omega}} = 0$$

and $\tau_{q\omega} = x_r^{[q]} - y_{r-1}^{[q]}$. Since $S_{q\omega}(1,0)$ is simply $y_0^{[q-1]} = 1$, we have

$$\lim_{q\to\infty} \frac{\log |\tau_{q\omega}|_v}{d^{q\omega}} = 0 = \lim_{q\to\infty} \frac{\log |S_{q\omega}(0,1)|_v}{d^{q\omega}},$$

which give us (4.4.3) and thus completes our proof.

□

4.3. **Proofs of the main theorems.** Now, we can show that the integral $\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |t-\beta|_v d\mu_{\varphi,v}$ can be computed by taking the limit of the average of $\log|\beta - w|_v$ on the points in $\varphi^{-k}(\alpha)$, as $k \to \infty$, for any nonexceptional point α.

Theorem 4.5. *Let $\alpha = [s : u]$ be a nonexceptional point in $\mathbb{P}^1(\overline{K})$. Then for any nonzero polynomial $F(t) \in \overline{K}[t]$ we have*

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log |F|_v \, d\mu_{\varphi,v} = \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ F(w)\neq 0}} \log |F(w)|_v.$$

where the $[w : 1]$ for which $\varphi^k([w : 1]) = \alpha$ are counted with multiplicity.

Proof. The polynomial F factors as $F(t) = \gamma \prod_{i=1}^{n}(t-\beta_i)$ where γ and $\beta_1, \dots, \beta_n$ are elements of $\overline{K}$. For each β_i, the multiplicity of β_i in $(\varphi^k)^*\alpha$ is at most $r((\varphi^k)^*\alpha)$ (where $r((\varphi^k)^*\alpha)$ is defined as in Section 3). Since α is not exceptional, we have $\lim_{k\to\infty} \frac{r((\varphi^k)^*\alpha)}{d^k} = 0$, by Lemma 3.3. Thus,

$$\lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ w\neq\beta_j}} \log|w-\beta_j|_v = \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ F(w)\neq 0}} \log|w-\beta_j|_v$$

for each β_j. Hence, it suffices to show that

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|t-\beta|_v \, d\mu_{\varphi,v} = \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ w\neq\beta}} \log|w-\beta|_v \tag{4.5.1}$$

for any $\beta \in \overline{K}$.

Note that $\varphi^k([w:1]) = [s:u]$ if and only if $uP_k(w,1) - sQ_k(w,1) = 0$. Thus, as polynomials in t, we have

$$uP_k(t,1) - sQ_k(t,1) = \eta_k \prod_{\varphi^k([w:1])=[s:u]} (t-w),$$

where $\eta_k \in \overline{K}$. We write

$$uP_k(t,1) - sQ_k(t,1) = (t-\beta)^{w_k} G_k(t,1)$$

for a polynomial G_k such that $G_k(\beta,1) \neq 0$, as in Proposition 4.3. Note that

$$G_k(t,1) = \eta_k \prod_{\substack{\varphi^k([w:1])=\alpha \\ w\neq\beta}} (t-w).$$

Plugging β in for t and taking logs of absolute values gives

$$\log|G_k(\beta,1)|_v = \log|\eta_k|_v + \sum_{\substack{\varphi^k([w:1])=[s:u] \\ w\neq\beta}} \log|w-\beta|_v. \tag{4.5.2}$$

Applying Proposition 4.3 therefore yields

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ w\neq\beta}} \log|w-\beta|_v + \frac{\log|\eta_k|_v}{d^k} \\ &\qquad = \lim_{k\to\infty} \frac{\log\max(|P_k(\beta,1)|_v, |Q_k(\beta,1)|_v)}{d^k}. \end{aligned} \tag{4.5.3}$$

Now, writing

$$uP_k(T_0,T_1) - sQ_k(T_0,T_1) = T_1^{w_k} V_k(T_0,T_1)$$

for some polynomial V_k such that $V_k(1,0) \neq 0$, we see that $\eta_k = V_k(1,0)$. Applying Proposition 4.3, we obtain

$$\lim_{k\to\infty} \frac{\log|\eta_k|_v}{d^k} = \lim_{k\to\infty} \frac{\log\max(|P_k(1,0)|_v, |Q_k(1,0)|_v)}{d^k}.$$

Substituting this equality into (4.5.3) gives

(4.5.4)

$$\lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ [w:1]\neq\beta}} \log|w-\beta|_v = \lim_{k\to\infty} \frac{\log\max(|P_k(\beta,1)|_v, |Q_k(\beta,1)|_v)}{d^k} - \lim_{k\to\infty} \frac{\log\max(|P_k(1,0)|_v, |Q_k(1,0)|_v)}{d^k}.$$

Using Definition 2.2, we obtain (4.5.1). □

Now, we show that the same result holds when we average $\log|\beta - w|_v$ over periodic points rather than inverse images of a point.

Theorem 4.6. *For any any polynomial $F \in \overline{K}[t]$ we have*

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|F|_v \, d\mu_{\varphi,v} = \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=[w:1] \\ F(w)\neq 0}} \log|F(w)|_v,$$

where the $[w:1]$ for which $\varphi^k([w:1]) = w$ are counted with multiplicity.

Proof. As in the proof of Theorem 4.5, it will suffice to show that

$$\int_{\mathbb{P}^1(\mathbb{C}_v)} \log|t-\beta|_v \, d\mu_{\varphi,v} = \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=[w:1] \\ w\neq\beta}} \log|w-\beta|_v \tag{4.6.1}$$

for any $\beta \in \overline{K}$ (this follows from the fact that the multiplicity of each β_i as a k-periodic point is bounded for all k by Proposition 4.4).

We have $\varphi^k([w:1]) = [w:1]$ if and only if $P_k(w,1) - wQ_k(w,1) = 0$. Thus,

$$P_k(t,1) - tQ_k(t,1) = \gamma_k \prod_{\varphi^k([w:1])=[w:1]} (t-w),$$

for some $\gamma_k \in \overline{K}$. We write

$$P_k(t,1) - tQ_k(t,1) = (t-\beta)^{n_k} H_k(t,1)$$

for a polynomial H_k such that $H_k(\beta,1) \neq 0$. We have

$$H_k(t,1) = \gamma_k \prod_{\substack{\varphi^k([w:1])=[w:1] \\ w\neq\beta}} (t-w).$$

Then, plugging β in for t, taking logs of absolute values, and applying Proposition 4.4 gives

$$\lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=[w:1] \\ w\neq\beta}} \log|\beta - w|_v + \frac{\log|\gamma_k|_v}{d^k} = \lim_{k\to\infty} \frac{\log\max(|P_k(\beta,1)|_v, |Q_k(\beta,1)|_v)}{d^k}. \tag{4.6.2}$$

Writing

$$T_1 P_k(T_0,T_1) - T_0 Q_k(T_0,T_1) = T_1^{n_k} W_k(T_0,T_1)$$

for a polynomial W_k such that $W_k(1,0)\neq 0$, we see that $\gamma_k = W_k(1,0)$. By Proposition 4.4, we have

$$\lim_{k\to\infty} \frac{\log|\gamma_k|_v}{d^k} = \lim_{k\to\infty} \frac{\log\max(|P_k(1,0)|_v, |Q_k(1,0)|_v)}{d^k}.$$

Combining this equality with (4.6.2) and Definition 2.2 yields (4.6.1). ☐

We are now ready to prove the results regarding the computation of the canonical height $h_\varphi(\beta)$. First, we'll need a lemma.

Lemma 4.7. *Let $\beta = [a:b]$ in $\mathbb{P}^1(\overline{K})$. Let $[a_1:b_1],\dots,[a_1:b_1]$ be the conjugates of $[a:b]$ under the action of $\mathrm{Gal}(\overline{K}/K)$. Then*

(4.7.1)

$$[K(\beta):K](\deg K) h_\varphi([a:b]) = \sum_{\textit{places } v \textit{ of } K} \lim_{k\to\infty} \sum_{i=1}^{n} \frac{\log\max(|P_k(a_i,b_i)|_v, |Q_k(a_i,b_i)|_v)}{d^k}.$$

Proof. For all but finitely many v, we have $|a_i|_v = |b_i|_v = 1$. Furthermore, for all but finitely many v, we have

$$\log\max(|P_k(s,t)|_v, |Q_k(s,t)|_v) = 0 \tag{4.7.2}$$

for all k whenever $|s|_v = |t|_v = 1$. This is true, for example, at all nonarchimedean v of good reduction for φ in the sense of [PST04]. Indeed, when v is a finite place, (4.7.2) will hold for all $|s|_v = |t|_v = 1$ unless either $|\operatorname{Res}(P(T_0,1),Q(T_0,1))|_v$ or $|\operatorname{Res}(P(1,T_1),Q(1,T_1))|_v$ is less than 1, where Res is the usual resultant of two polynomials (see [BK86, p. 279, Proposition 4]). Thus, we can interchange the limit and the sum on the right-hand side of (4.7.1) so that

$$\begin{aligned}\lim_{k\to\infty} \sum_{\text{places } v \text{ of } K} \sum_{i=1}^{n} \frac{\log\max(|P_k(a_i,b_i)|_v, |Q_k(a_i,b_i)|_v)}{d^k} \\ = \sum_{\text{places } v \text{ of } K} \lim_{k\to\infty} \sum_{i=1}^{n} \frac{\log\max(|P_k(a_i,b_i)|_v, |Q_k(a_i,b_i)|_v)}{d^k}.\end{aligned} \tag{4.7.3}$$

Now, let L be the field $K(\beta)$ and let w be a place of L that extends the place v of K; we write $w \mid v$. The field L has n embeddings $i : L \hookrightarrow \mathbb{C}_v$; for exactly $[L_w : K_v]$ of these embeddings, we have $|i(x)|_v = |x|_w$ for all $x \in L$. This yields $[L_w : K_v]$ conjugates $[a' : b']$ of $[a : b]$ such that $|a|_w = |a'|_v$ and $|b|_w = |b'|_w$. Hence, we see that

$$\begin{aligned}&\sum_{i=1}^{n} \log\max(|P_k(a_i,b_i)|_v, |Q_k(a_i,b_i)|_v) \\ &\quad = \sum_{w|v} [L_w : K_v] \log\max(|P_k(a,b)|_w, |Q_k(a,b)|_v).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\sum_{\text{places } v \text{ of } K} \sum_{i=1}^{n} \log\max(|P_k(a_i,b_i)|_v, |Q_k(a_i,b_i)|_v) \\ &\quad = [K(\beta) : K](\deg K) h(\varphi^k([a : b])),\end{aligned}$$

by (1.0.5). It follows from (1.0.6) and (4.7.3) that we therefore have

$$\begin{aligned}&\sum_{\text{places } v \text{ of } K} \lim_{k\to\infty} \sum_{i=1}^{n} \frac{\log\max(|P_k(a_i,b_i)|_v, |Q_k(a_i,b_i)|_v)}{d^k} \\ &\quad = [K(\beta) : K](\deg K) \lim_{k\to\infty} \frac{h(\varphi^k([a : b]))}{d^k} \\ &\quad = [K(\beta) : K](\deg K) h_\varphi([a : b]).\end{aligned}$$

□

Theorem 4.8. *Let α be any point in $\mathbb{P}^1(\overline{K})$ that is not an exceptional point of φ. Then, for any $\beta \in \overline{K}$ and any nonzero irreducible $F \in K[t]$ such that $F(\beta) = 0$, we have*

$$\begin{aligned}&(\deg K)(\deg F)(h_\varphi(\beta) - h_\varphi(\infty)) \\ &\quad = \sum_{\text{places } v \text{ of } K} \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ F(w)\neq 0}} \log|F(w)|_v,\end{aligned}$$

where the $[w : 1]$ for which $\varphi^k([w : 1]) = \alpha$ are counted with multiplicity.

Proof. Write $F(t) = \gamma \prod_{i=1}^n (t - \beta_i)$ where $\gamma \in K$ and the β_i are the conjugates of β under the action of $\mathrm{Gal}(\overline{K}/K)$. By the product formula, we have $\sum_{\text{places } v \text{ of } K} |\gamma|_v = 0$. Thus, using Theorem 4.5 and Definition 2.2, we see that

$$
\begin{aligned}
&\sum_{\text{places } v \text{ of } K} \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ F(w)\neq 0}} \log |F(w)|_v \\
&= \sum_{\text{places } v \text{ of } K} \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=\alpha \\ F(w)\neq 0}} \log \left| \prod_{i=1}^n (w - \beta_i) \right|_v \\
&= \sum_{i=1}^n \lim_{k\to\infty} \frac{\log \max(|P_k(\beta_i, 1)|_v, |Q_k(\beta_i, 1)|_v)}{d^k} \\
&- (\deg F) \lim_{k\to\infty} \frac{\log \max(|P_k(1, 0)|_v, |Q_k(1, 0)|_v)}{d^k}.
\end{aligned} \tag{4.8.1}
$$

By Lemma 4.7, the quantity on the last two lines is equal to

$$(\deg F)(\deg K)(h_\varphi(\beta) - h_\varphi(\infty)),$$

as desired. □

Theorem 4.9. *For any $\beta \in \overline{K}$ and any nonzero irreducible $F \in K[t]$ such that $F(\beta) = 0$, we have*

$$
\begin{aligned}
&(\deg K)(\deg F)(h_\varphi(\beta) - h_\varphi(\infty)) \\
&\qquad = \sum_{\textit{places } v \textit{ of } K} \lim_{k\to\infty} \frac{1}{d^k} \sum_{\substack{\varphi^k([w:1])=[w:1] \\ F(w)\neq 0}} \log |F(w)|_v,
\end{aligned}
$$

where the $[w : 1]$ for which $\varphi^k([w : 1]) = w$ are counted with multiplicity.

Proof. The proof is the same as the proof of Theorem 4.8, using Theorem 4.6 in place of Theorem 4.5. □

5. A counterexample

The main theorems of this paper are *not* true when we work over the complex numbers $\mathbb{C}$ rather than $\overline{K}$. Let $K = \mathbb{Q}$ and let $\varphi([x : y]) = [x^2 : y^2]$ be the usual squaring map. Let v be the archimedean place of $\mathbb{Q}$, so that $\mathbb{C}_v$ is just the usual complex numbers $\mathbb{C}$. We define the function ψ on the positive integers recursively by $\psi(1) = 2$ and $\psi(n) = 2^{(n\psi(n-1))}$. Let $\alpha = \sum_{n=1}^\infty 1/\psi(n)$ and let $\beta = e^{2\pi i \alpha}$. Note that for any t, we have $|e^{2\pi it} - 1| \leq \pi(t - [[t]])$, (where $[[t]]$ is the greatest integer less than or equal to t). Letting $\ell_n = \log_2 \psi(n)$, we then have

$$\begin{aligned}&\frac{1}{2^{\ell_n}} \sum_{w^{2^{\ell_n}}=1} \log|w-\beta|_v = \frac{\log|\beta^{\psi(n)}-1|}{\psi(n)}\\ &\leq \frac{1}{\psi(n)} \log(\pi\left(\psi(n)\alpha - [[\psi(n)\alpha]]\right))\\ &\leq \frac{1}{\psi(n)} \log\left(\pi \frac{\psi(n)}{\psi(n+1)} \sum_{j=0}^{\infty} \frac{1}{2^{j\psi(n+1)}}\right)\\ &\leq \log\pi + 1 - n\log 2 + \log 2.\end{aligned}$$

Thus, $\frac{1}{2^{\ell_n}} \sum_{w^{2^{\ell_n}}=1} \log|\beta - w|_v$ goes to $-\infty$ as $n \to \infty$, so

$$\lim_{k\to\infty} \frac{1}{2^k} \sum_{w^{2^k}=1} \log|w-\beta|_v$$

does not exist.

6. Applications and further questions

6.1. **Lyapunov exponents.** The Lyapunov exponent of a rational map $\varphi : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ is defined as

$$L(\varphi) = \int_{\mathbb{P}^1(\mathbb{C})} \log\|D\varphi\| d\mu_\varphi,$$

where D is the usual derivative, $\|\cdot\|$ is any metric on $\mathbb{P}^1$, and μ_φ is the unique measure of maximal entropy measure for φ on $\mathbb{P}^1$; this measure of maximal entropy is the same as the Brolin-Lyubich measure discussed in Section 2. Choosing coordinates $[T_0 : T_1]$ for $\mathbb{P}^1_{\mathbb{C}}$, letting $t = T_0/T$, and writing $\varphi(t) = P(t)/Q(t)$ for polynomials P and Q, we have

$$L(\varphi) = \int_{\mathbb{P}^1(\mathbb{C})} \log|\varphi'| d\mu_\varphi.$$

The Lyapunov exponent can be computed via equidistribution on certain subsequences of inverse images of nonexceptional points in $\mathbb{P}^1(\mathbb{C})$ (see [DeM03], [Mañ88]). That is, given a nonexceptional point α in $\mathbb{P}^1(\mathbb{C})$, there is an infinite strictly increasing sequence of integers $(m_i)_{i=1}^\infty$ such that

$$L(\varphi) = \lim_{i\to\infty} \frac{1}{(\deg\varphi)^{m_i}} \sum_{\substack{\varphi^{m_i}(\beta)=\alpha\\ \varphi'(\beta)\neq 0\\ \beta\neq\infty}} \log|\varphi'(\beta)|.$$

It is not known, however, if $L(\varphi)$ can be computed by taking the limit of the average φ' on the periodic points of φ.

When φ is defined over a number field K, however, we obtain the following result as a corollary of Theorem 4.6.1.

Corollary 6.1. *Let K be a number field and let $\varphi : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ be a nonconstant rational map that is defined via base extension from a map $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$. Let φ' be defined as above. Then*

$$L(\varphi) = \lim_{k\to\infty} \frac{1}{(\deg\varphi)^k} \sum_{\substack{\varphi^k(\xi)=\xi \\ \varphi'(\xi)\neq 0 \\ \xi\neq\infty}} \log|\varphi'(\xi)|.$$

Proof. We may write φ' as a quotient of polynomials $A(t)/B(t)$ with coefficients in K. This yields $\log|\varphi'(t)| = \log|A(t)| - \log|B(t)|$. The corollary then follows immediately from Theorem 4.6. ☐

This corollary says that the Lyapunov exponent of a rational function φ defined over a number is completely determined by the derivative of φ at the periodic points of φ. This means that the derivative of φ at the periodic points of φ also determines the Hausdorff dimension of the Julia set (see [FLM83]).

6.2. **Symmetry of canonical heights.** In [ST], we show that when ∞ is not in the v-adic Julia set of φ for any archimedean v, we have

$$\lim_{k\to\infty} \frac{1}{d^k} \sum_{\varphi^k([w:1])=[w:1]} h(w) = \lim_{\ell\to\infty} \frac{1}{2^\ell} \sum_{\xi^{2^\ell}=\xi} h_\varphi(\xi). \tag{6.1.1}$$

This can be thought of as a symmetry relation, connecting h of the φ-periodic points with h_φ of the roots of unity. The proof uses Theorem 4.9 along with Lyubich's equidistribution theorem ([Lyu83]) and some adelic intersection theory (see [Zha95] and [Zha92]). We are also able to use Theorem 4.9 to prove that

$$h_\varphi(\beta) - h(\beta) \leq \lim_{k\to\infty} \frac{1}{d^k} \sum_{\varphi^k([w:1])=[w:1]} h(w) + h_\varphi(\infty) + \log 2.$$

Our proof of (6.1.1) does not work when ∞ is in the v-adic Julia set of φ, for in that case the local height $\hat{h}_v$ is not bounded on the v-adic Julia set. Unfortunately, the v-adic Julia set is all of $\mathbb{P}^1(\mathbb{C}_v)$ when v is archimedean for many of the most interesting rational maps φ. This is the case, for example, when φ is the map obtained by taking the multiplication-by-2 map on an elliptic curve and modding out by the hyperelliptic involution (such a map is called a Lattès map).

On the other hand, the usual local height $\hat{h}_v(t)$ of an element $t \in \mathbb{C}_v$ is simply $\max(\log|t|_v, 0)$, which is only a little bit different from $\log|t|_v$, and Theorem 4.6 proves a suitable equidistribution theorem for $\log|t|_v$. We hope to extend the techniques of this paper so that we can prove an analog of Theorem 4.6 for functions such as $\max(\log|t|_v, 0)$.

6.3. **Computing with points of small height.** The results in [Bil97], [Aut01], [BR05], [FRL04b], [FRL04a], and [CL04] all apply not only to the periodic points and backwards iterates of a point that we treat in this paper but to all points of small height in the algebraic closure of a number field K. For example, one the main theorems in [BR05], [FRL04b], [FRL04a], and [CL04] states that for any continuous function g on $\mathbb{P}^1(\mathbb{C}_v)$ and any infinite nonrepeating sequence of points (α_n) in $\mathbb{P}^1(\overline{K})$ such that $\lim_{n\to\infty} h_\varphi(\alpha_n) = 0$, one has

$$\lim_{n\to\infty} \frac{1}{|\operatorname{Gal}(\alpha_n)|} \sum_{\sigma\in\operatorname{Gal}(\alpha_n)} g(\alpha_n^\sigma) = \int_{\mathbb{P}^1(\mathbb{C}_v)} g\, d\mu_{v,\varphi}, \tag{6.1.2}$$

where $\operatorname{Gal}(\alpha_n)$ is the Galois group of the Galois closure of $K(\alpha_n)$ over K.

Baker, Ih, and Rumely ([BIR05]) and Autissier ([Aut]) have produced counterexamples that show that (6.1.2) does not always hold when the function g is replaced with $\log|F|_v$ for F a polynomial. All of these examples involve infinite nonrepeating sequences of points $(\alpha_n) \in \bar{\mathbb{Q}}$ such that $\lim_{n\to\infty} h(\alpha_n) = 0$ and

$$\lim_{n\to\infty} \frac{1}{|\operatorname{Gal}(\alpha_n)|} \sum_{\sigma\in\operatorname{Gal}(\alpha_n)} \log|\alpha_n^\sigma - 2| \neq \int_0^1 \log|e^{2\pi i\theta} - 2| d\theta.$$

The points (α_n) are not preperiodic in any of these examples Thus, it may be possible to prove that the main results of this paper continue to hold when we work with any nonrepeating sequence of Galois orbits of preperiodic points. This would imply the following conjectured generalization of Siegel's theorem for integral points.

Conjecture 6.2 (Ih). *For any nonpreperiodic point $\beta \in \mathbb{P}^1(\overline{K})$, there are at most finitely many β-integral preperiodic points of φ in $\mathbb{P}^1(\overline{K})$. (Here, α is said to be β-integral if the Zariski closure of α does not meet the Zariski closure of β in $\mathbb{P}^1_{\mathcal{O}_K}$, where $\mathcal{O}_K$ is the ring of integers of K.)*

Baker, Ih, and Rumely have proven that this is true when φ is a Lattès map or the usual squaring map $x \mapsto x^2$. Using Theorem 4.9 and arguing as in [BIR05], it is possible to derive the following weak version of Ih's conjecture in general.

Proposition 6.3. *For any nonpreperiodic point $\beta \in \mathbb{P}^1(\overline{K})$, there are at most finitely many n such that all $\alpha \in \mathbb{P}^1(\overline{K})$ of period n are β-integral.*

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