Asymptotic isoperimetrics of sublinear wedge domains

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In this note we consider the isoperimetric problem for **(i)** a domain Ω in $\mathbb{R}^2$ described by

$$\Omega = \{(x,y) :\ 0 < y < F(x),\ x > 0\},$$

where $F(x)$ is positive on $\{x > 0\}$, continuous on $\{x \geq 0\}$, such that

$$F(x) \text{ is increasing to } \infty, \quad F(x)/x \text{ is decreasing to } 0, \qquad \text{as } x \to +\infty,$$

and **(ii)** isoperimetric profile $\mathcal{I}(v)$, of Ω, $v > 0$, defined by

$$\mathcal{I}(v) = \inf\{L(\partial G \cap \Omega) :\ A(G) = v\},$$

where G varies over all subdomains G of $\overline{\Omega}$ with compact closure, and $\partial G \cap \Omega$ is piecewise C^1 . That is, in calculating the isoperimetric profile we only consider that part of ∂G in the *interior* of Ω. Written as an inequality, one has

$$L(\partial G \cap \Omega) \geq \mathcal{I}(A(G)).$$

The classical example, of considering only that part of the boundary of G which is in the interior of Ω, is the case where $\Omega = (\mathbb{R}^2)^+$, the Euclidean upper half plane, in which case

$$\mathcal{I}(v) = \sqrt{2\pi}v^{1/2}.$$

Here, for each $v > 0$, we can identify a minimizing domain G_o for which $\mathcal{I}(v) = L(\partial G_o \cap \Omega)$, namely, the upper circular semi-disk of area v with diameter on the x–axis. Furthermore, the semi-disk is unique among all competing domains G.

More generally, for the second isoperimetric problem, consider the wedge of angle $\alpha \in (0, \pi]$, defined by the x–axis and the line

$$y = (\tan\,\alpha)x, \qquad x > 0.$$

Then, for any $v > 0$, the minimizing domain for $\mathcal{I}(v)$ is bounded by the wedge and the circular arc of radius $\sqrt{2v/\alpha}$.

Here, we will not be able to identify subdomains which actually minimize the functional $\mathcal{I}(v)$, but we will be able to describe the asymptotics of the function $\mathcal{I}(v)$ as $v \to +\infty$.

Theorem. *Consider the function*

$$v(x) := \int_0^x F(t)\,dt$$

and its inverse function $x = \Phi(v)$. *Then*

$$\mathcal{I}(v) \sim F\circ\Phi(v) \qquad \text{as} \;\; v \to +\infty.$$

The geometric content of the theorem is as follows: For any $x_o > 0$, set

$$\begin{aligned} \Omega_{x_o} &= \{(x,y):\ 0 < y < F(x),\ 0 < x < x_o\}, \\ \Pi_{x_o} &= \{(x_o,y):\ 0 \le y \le F(x_o)\}. \end{aligned}$$

So Ω_{x_o} is the subdomain of Ω bounded by the x–axis, y–axis (if $F(0) > 0$), the graph of $y = F(x)$, and the line $x = x_o$. Then, by definition,

$$\mathcal{I}(v(x_o)) \le L(\Pi_{x_o}) = F(x_o).$$

The theorem states that, as $v \to \infty$, the domains $\Omega_{\Phi(v)}$ are increasingly precise approximations to minimizers of the isoperimetric problem for $\mathcal{I}(v)$, achieving precision in the limit as $v \to \infty$.

Note that the theorem is not true if we do not have $F(x) = o(x)$ as $x \to \infty$. Indeed, consider the wedge described by the line

$$F(x) = (\tan\,\alpha)x, \qquad x > 0,$$

described above. Then for R given by $v = \alpha R^2/2$, the minimizer for $\mathcal{I}(v)$ is given by the arc of radius R centered at the origin. But $x = \Phi(v)$ satisfies $v = (\tan\,\alpha)x^2/2$, which implies

$$x = \sqrt{\frac{\alpha}{\tan\,\alpha}}R,$$

and

$$\begin{aligned} &\mathcal{I}(v) = R\alpha, \quad F(x) = \sqrt{\alpha \tan\alpha}R \qquad \Rightarrow \\ &F(\Phi(v))/\mathcal{I}(v) = \sqrt{\frac{\tan\,\alpha}{\alpha}} > 1 \qquad \forall\, v > 0, \end{aligned}$$

which contradicts the claim of the theorem. On the other hand, it is not clear at this juncture whether one can do away with the assumption that $F(x)/x$ is decreasing. Also note that if $F(x)$ is C^2 and concave, the minimizing domains consist of domains bounded by the x–axis, the graph of $y = F(x)$, and arcs of circles intersecting the two sides of the wedge orthogonally. One can calculate them explicitly and verify the truth of the Theorem.

Background. The result was motivated by a query of I. Benjamini, for which a partial answer was given by P. Pansu [7]. The question was to consider

$$\Omega^* := \Omega \times \mathbb{R} \subset \mathbb{R}^3 = \{(x,y,z) \in \mathbb{R}^3 : \ 0 < y < F(x), \ x > 0, \ z \in \mathbb{R}\},$$

and the subdomains

$$\Omega^*(R) := \{x^2 + z^2 < R^2\} \cap \Omega^*,$$

and ask whether this family $\Omega^*(R)$ of subdomains of Ω^* converged, asymptotically, to "isoperimetric minimizers" as $R \to \infty$ (in the sense of our Theorem, here). Pansu showed that if $v = V(\Omega^*(R))$ (V denotes volume), with inverse function $R = \Psi(v)$ and isoperimetric profile $\mathcal{I}(v)$ for subdomains of Ω^*, then

$$cA(\partial\Omega^*(\Psi(v))) \le \mathcal{I}(v) \le A(\partial\Omega^*(\Psi(v))) \qquad \text{for some} \ \ c \in (0,1),$$

as $v \to \infty$. The conjecture is that $A(\partial\Omega^*(\Psi(v))) \sim \mathcal{I}(v)$ as $v \to \infty$. So one might view the Theorem, here, as a tentative indication for the truth of the conjecture in Benjamini's problem.

For background and more general setting of isoperimetric inequalities in Riemannian geometry, see the introductory articles: Osserman [5], [6], Treibergs [8], Gromov [3], and the books: Gromov [4], Chavel [1], [2].

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Lemma. *We have*

$$\lim_{v\to\infty} \frac{\mathcal{I}^2(v)}{v} = 0. \tag{1}$$

Proof. Since $F(x)/x$ is decreasing, we have

$$\frac{F(x/2)}{x/2} \ge \frac{F(x)}{x} \qquad \Rightarrow \qquad F(x) \le 2F(x/2),$$

which implies, for the domain Ω_x,

$$\frac{\mathcal{I}^2(v)}{v} \le \frac{F^2(x)}{\int_0^x F} \le \frac{F^2(x)}{\int_{x/2}^x F} \le \frac{F^2(x)}{(x/2)F(x/2)} \le 4\,\frac{F(x)}{x} \to 0,$$

which is the claim.

Proof of the Theorem. Assume that G is relatively compact in Ω. Then

$$L^2(\partial G \cap \Omega) = L^2(\partial G) \ge 4\pi A(G)$$

by the usual classical isoperimetric inequality. So such subdomains of Ω cannot be approximate minimizers of $\mathcal{I}(v)$ for large v.

We henceforth consider subdomains G of Ω with $\partial G \cap \partial \Omega \neq \emptyset$. For any such G, let G_1 denote the convex hull of G in $\mathbb{R}^2$, and $G_2 = G_1 \cap \Omega$. Then

$$A(G_2) \geq A(G) \quad \text{and} \quad L(\partial G_2 \cap \Omega) \leq L(\partial G \cap \Omega).$$

Let G_3 denote the unbounded component of $\Omega \setminus G_2$, and $G_4 = \Omega \setminus G_3$. Then

$$A(G_4) \geq A(G) \quad \text{and} \quad L(\partial G_4 \cap \Omega) \leq L(\partial G \cap \Omega),$$

so

$$L^2(\partial G \cap \Omega)/A(G) \geq L^2(\partial G_4 \cap \Omega)/A(G_4)$$

We distinguish three cases.

First. $(\partial G_4 \cap \Omega)$ consists of a convex curve both of whose endpoints lie on the x–axis. Then

$$L^2(\partial G_4 \cap \Omega) \geq 2\pi A(G_4),$$

as noted earlier; so such subdomains of Ω cannot be approximate minimizers of $\mathcal{I}(v)$ for large v.

Second. $(\partial G_4 \cap \Omega)$ consists of a convex curve Γ, whose pair of endpoints intersect the graph of $y = F(x)$. So assume that Γ is the graph of a convex function $y = \phi(x)$, $x_1 < x < x_2$. Let y^* be defined by

$$y^* = \phi(x^*) := \min\{\phi(x) : x \in [x_1, x_2]\} \leq F(x_1) < F(x_2).$$

Consider the polygonal path Γ^* given by the line segment from $(x_1, F(x_1))$ to (x^*, y^*) followed by the line segment from (x^*, y^*) to $(x_2, F(x_2))$. Standard calculus implies

$$\begin{aligned} L^2(\Gamma^*) &\geq (x_2 - x_1)^2 + \{F(x_2) + F(x_1) - 2y^*\}^2 \\ &\geq (x_2 - x_1)^2 + \{F(x_2) - y^*\}^2, \end{aligned}$$

which implies

$$\begin{aligned} \frac{L^2(\Gamma)}{A(G)} \geq \frac{L^2(\Gamma^*)}{A(G)} &\geq \frac{(x_2 - x_1)^2 + \{F(x_2) + F(x_1) - 2y^*\}^2}{\{F(x_2) - y^*\}(x_2 - x_1)} \\ &\geq \frac{x_2 - x_1}{F(x_2) - y^*} + \frac{F(x_2) - y^*}{(x_2 - x_1)}. \end{aligned}$$

That is,

$$\frac{L^2(\Gamma)}{A(G)} \geq \frac{x_2 - x_1}{F(x_2) - y^*} + \frac{F(x_2) - y^*}{(x_2 - x_1)} \geq 2,$$

which implies such a G cannot be an approximate minimizer.

Third. $\partial G_4 \cap \Omega$ consists of a curve connecting the graph of $y = F(x)$ with the x–axis.

We first consider what happens when $G_4 \supset G$. Consider the translation $\tau_\beta : (x, y) \mapsto (x - \beta, y)$, leftward along the x–axis of distance $\beta > 0$. Set

$$\mathcal{G}_\beta = \tau_\beta(G_4) \cap \Omega.$$

Then $A(\mathcal{G}_\beta)$ decreases strictly and continuously as β increases, and $L(\partial\mathcal{G}_\beta \cap \Omega)$ decreases as β increases (the decrease of both A and L a consequence of the monotonicity of $F(x)$), which implies

$$A(\mathcal{G}_\beta) < A(G_4), \quad L(\partial\mathcal{G}_\beta \cap \Omega) \le L(\partial G \cap \Omega) \quad \text{for all} \quad \beta > 0,$$

and there exists an $\beta_o > 0$ so that $A(\mathcal{G}_{\beta_o}) = A(G)$. This will imply $L(\partial\mathcal{G}_{\beta_o} \cap \Omega) \le L(\partial G \cap \Omega)$. So for $v = A(G)$ we may substitute G_4 for G. Said differently, we may assume that ∂G consists of a curve Γ connecting the graph of $y = F(x)$ with the x–axis.

For any such G, let $v = A(G)$, $x_o = \Phi(v)$ defined by

$$v = \int_0^{x_o} F(x)\, dx,$$

and

$$x_1 = \min\{x : \ (x, y) \in \Gamma\}, \qquad x_2 = \max\{x : \ (x, y) \in \Gamma\},$$

$$x_2 = (1 + \gamma)x_1, \qquad x_2 - x_1 = \gamma x_1, \qquad 0 \le \gamma \le 1.$$

If Ω_{x_o} is not a minimizer for v, and $L(\Gamma) < L(\Pi_{x_o}) = F(x_o)$, then

$$F(x_2) \ge F(x_o) \ge L(\Gamma) \ge x_2 - x_1 = \frac{\gamma}{1+\gamma} x_2,$$

which implies

$$\frac{F(x_2)}{x_2} \ge \frac{\gamma}{1+\gamma};$$

therefore, as $v \to \infty$, we have $x_o \to \infty$, which implies $x_2 \to \infty$, which implies

(2) $$\lim_{v\to\infty} \gamma = 0, \qquad \lim_{v\to\infty} \frac{x_2}{x_1} = 1,$$

which implies

$$\lim_{v\to\infty} x_1 = \infty.$$

Since $F(x)/x$ decreases to 0 as $x \to \infty$, we have $1 \le F(x_2)/F(x_1) \le x_2/x_1 \to 1$ as $v \to \infty$, so

(3) $$\lim_{v\to\infty} \frac{F(x_2)}{F(x_1)} = 1.$$

If Ω_{x_o} is not a minimizer, and $L(\Gamma) < F(x_o)$, then one endpoint of Γ must be at $(x_1, F(x_1))$ and the other at $(x_2, 0)$, $x_1 < x_o < x_2$. Therefore, by projecting Γ onto Π_{x_1}, we have

$$F(x_1) \le L(\Gamma).$$

Suppose we are given a constant ε and a sequence $v_j \to \infty$ for which

$$\mathcal{I}(v_j) \leq \varepsilon F(\Phi(v_j)), \qquad 0 < \varepsilon < 1.$$

Then we have a sequence of domains G_j, and a constant $\delta \in (\varepsilon, 1)$ so that

$$A(G_j) = v_j \text{ and } L(\Gamma_j) \leq \delta F(x_{o,j}) \qquad \text{for all } j.$$

This implies

$$F(x_{1,j}) \leq L(\Gamma_j) \leq \delta F(x_{o,j}) \leq \delta F(x_{2,j}) \qquad \text{for all } j,$$

which implies $F(x_{2,j})/F(x_{1,j})$ is bounded away from 1, which is a contradiction. So $\mathcal{I}(v) \sim F(\Phi(v))$ as $v \to \infty$, which is the theorem.

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