Large time decay of the heat kernel of λ–transient Riemannian manifolds

Isaac Chavel
The City College of the
City University of New York
New York, NY 10031

Edgar A. Feldman
Graduate School of the
City University of New York
New York, NY 10036

Leon Karp
Lehman College of the
City University of New York
New York, NY 10468

Let M be a noncompact Riemannian manifold with Laplace–Beltrami operator Δ acting on functions on M, $\lambda =: \lambda(M) \geq 0$ the bottom of $\operatorname{spec}(-\Delta)$, and $p(x,y,t)$ (where (x,y,t) is an element of $M \times M \times (0,+\infty)$) the attendant minimal positive heat kernel. In this note we prove the following

Theorem. *Assume that*

$$\int_0^{+\infty} e^{\lambda t} p(x,y,t)\, dt < +\infty \tag{1}$$

for some pair (x,y), $x \neq y$. *Then*

$$e^{\lambda t} p(x,y,t) = o(t^{-1}) \tag{2}$$

as $t \uparrow +\infty$, *uniformly on compact subsets of* $M \times M$.

The inequality (1) is usually referred to as *λ–transience of M*. See [5]; also see [3]. We add that in [3] it is shown that, in general, $e^{\lambda t} p$ always has a limit when $t \uparrow +\infty$, which either is identically equal to 0 on all of $M \times M$ or never vanishes on all of $M \times M$. So our discussion here is within the context of the first of the two alternatives.

The result (2) is best possible in the sense that in $\mathbf{R}^2$ one has $\lambda = 0$, $p(x,x,t) = 1/4\pi t$, and the integral in (1) diverges.

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The result is sharp, in the sense that that one can construct (see Benjamini [1]) a surface of revolution which is 0–transient and whose heat kernel satisfies

$$p(o,o,t) \geq \frac{\text{const.}}{t\ln^2 t}$$

where o is the pole of the surface, for large t. (See Benjamini's paper for other examples.)

We note that if M is complete noncompact with bounded geometry (that is, Ricci curvature bounded from below and strictly positive injectivity radius), then it is a general result of Chavel–Feldman [2], improving an earlier result of Varopoulos [6], that

$$p(x,y,t) = O(t^{-1/2})$$

as $t \uparrow +\infty$.

We also note that the semigroup property and the Cauchy–Schwarz inequality imply

$$p(x,y,t) \leq \sqrt{p(x,x,t)}\sqrt{p(y,y,t)};$$

so it suffices to prove (2) for $x = y$.

Lemma. *For all $x \in M$ we have*

$$e^{\lambda t} p(x,x,t)$$

is a decreasing function of t.

Proof. Let D be a relatively compact domain in M with smooth boundary, and Dirichlet heat kernel q. Then one has $e^{\lambda t}q(x,x,t)$ is a decreasing function of t from the Sturm–Liouville eigenvalue–eigenfunction expansion of q.

Now pick an exhaustion of M, $D_j \uparrow M$ as $j \uparrow +\infty$, by domains which are relatively compact in M and which possess smooth boundary. Let q_j denote the Dirichlet heat kernel of D_j. It is standard that

$$q_j \uparrow p.$$

The lemma follows immediately. □

The lemma seems to have been known for some time. Peter Li has shown us (private communication — in which he derives the lemma with a cut–off function argument instead of our argument) that one can easily deduce from the lemma and from the monotonicity of $\lambda_1(D)$ with respect to D ($\lambda_1(D)$ is the lowest eigenvalue of D for D relatively compact) that

$$\lim_{t \uparrow +\infty} \frac{\ln p(x,y,t)}{t} = -\lambda.$$

See [4]. For recent improvements see [3].

Proof of the Theorem. It is standard (see [5]) that if (1) is valid for a given pair (x, y), $x \neq y$, then (1) is valid for all $x \neq y$. Furthermore, if for any fixed x we set

$$\mathcal{D}_x(y) = \int_1^{+\infty} e^{\lambda t} p(x, y, t)\, dt \qquad y \in M,$$

then $\mathcal{D}_x \in L^1_{loc}(M)$.

Now consider $\Delta \mathcal{D}_x$ as a distribution. Given $\varphi \in C_c^\infty(M)$ we have

$$\begin{aligned}
(\Delta \mathcal{D}_x) \cdot \varphi &= \int \mathcal{D}_x(y)(\Delta\varphi)(y)\, dV(y) \\
&= \lim_{T \uparrow +\infty} \int_1^T e^{\lambda t}\, dt \int_M p(x, y, t)(\Delta\varphi)(y)\, dV(y) \\
&= \lim_{T \uparrow +\infty} \int_1^T e^{\lambda t}\, dt \int_M (\Delta_y p)(x, y, t)\varphi(y)\, dV(y) \\
&= \lim_{T \uparrow +\infty} \left\{ \int_M e^{\lambda T} p(x, y, T)\varphi(y)\, dV(y) \right. \\
&\qquad - \int_M e^{\lambda} p(x, y, 1)\varphi(y)\, dV(y) \\
&\qquad \left. - \lambda \int_1^T e^{\lambda t}\, dt \int_M p(x, y, t)\varphi(y)\, dV(y) \right\} \\
&= -\lambda \mathcal{D}_x \cdot \varphi - \int_M e^{\lambda} p(x, y, 1)\varphi(y)\, dV(y),
\end{aligned}$$

by the λ–transience of M (see (5.5) of [5]). Therefore,

$$\Delta \mathcal{D}_x + \lambda \mathcal{D}_x = e^{\lambda} p(x,\ \ , 1) \tag{3}$$

in the sense of distributions. But the right hand side of (3) is C^∞ on all of M. Thus the distribution $\mathcal{D}_x$ may be redefined on a set of measure 0 to become a C^∞ function $\mathcal{K}$ on M. Now approach x along a sequence of points for which $\mathcal{D}_x$ and $\mathcal{K}$ agree. Since the values of $\mathcal{K}$ are bounded on the sequence, one concludes via Fatou's lemma that

$$\int_1^{+\infty} e^{\lambda t} p(x, x, t)\, dt < +\infty.$$

Since $e^{\lambda t} p(x, x, t)$ decreases with respect to t, (2) follows.

The uniformity of the convergence on compact subsets also follows from the monotonicity of $e^{\lambda t} p(x, x, t)$. □

References

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