THE NEUMANN-CHEEGER CONSTANT OF THE JUNGLE GYM

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Our "jungle gyms" are 2–dimensional differentiable manifolds M, with preferred Riemannian metrics, associated to graphs. Our interest is in proving that the validity of the Neumann–Cheeger inequality on any graph under consideration extends to one on the associated jungle gym. We have already considered this matter in (1996), in the general context of the invariance of volume doubling and Neumann–Poincaré inequalities under rough isometries, and have shown there how the general result (Theorem 4) applies to the examples that follow. Here our approach to these examples is explicitly geometric and quite elementary — the most advanced result is the Dido isoperimetric inequality. So it is the concrete geometry of these surfaces that attracts us to this exercise.

Our original motivation for studying these examples, as general as our collection of graphs, was the search for examples of Riemannian manifolds of polynomial volume growth for which we have upper and lower bounds on the minimal positive heat kernel $p(x, y, t)$ (where x and y vary over M and $t > 0$), the bounds given by

$$\text{const.}t^{-\nu/2}e^{-d^2(x,y)/\text{const.}t} \leq p(x, y, t) \leq \text{const.}t^{-\nu/2}e^{-d^2(x,y)/\text{const.}t}, \tag{0.1}$$

for large $t > 0$. See the general background discussion in (1996), in particular, Theorem 3.

1. The graphs and their jungle gyms

Suppose we are given a countable set $\mathcal{G}$, such that to each $\xi \in \mathcal{G}$ we have a finite nonempty subset $\mathsf{N}(\xi) \subseteq \mathcal{G} \setminus \{\xi\}$, of cardinality $m(\xi)$, each element of which is referred to as a *neighbor of* ξ. Furthermore, we require that $\eta \in \mathsf{N}(\xi)$ if and only if $\xi \in \mathsf{N}(\eta)$. Then one determines a

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graph structure $\mathbf{G}$ by postulating the existence of precisely one oriented edge from any ξ to each of its neighbors, that is, the elements of $\mathsf{N}(\xi)$. For convenience, we henceforth make no distinction between $\mathbf{G}$ and its underlying set $\mathcal{G}$. We refer to $m(\xi)$ as the *valence of* $\mathbf{G}$ *at* ξ. We say that the graph $\mathbf{G}$ has *bounded geometry* if the valence function $m(\xi)$ is bounded uniformly from above on all of $\mathbf{G}$. A sequence of points $(\xi_0, \dots, \xi_k)$ is a *combinatorial path of length* k if $\xi_j \in \mathsf{N}(\xi_{j-1})$ for all $j = 1, \dots, k$. The graph $\mathbf{G}$ is called *connected* if any two points are connected by a path. Note that $m(\xi) \geq 1$ for all ξ if $\mathbf{G}$ is connected. For any two vertices ξ and η in the connected graph $\mathbf{G}$, one defines their *distance* $\mathsf{d}(\xi, \eta)$ to be the infimum of the length of all paths connecting ξ to η. We also refer to d as the *combinatorial metric.* We denote the metric "disks" by

$$\beta(\xi; k) = \{\eta \in \mathbf{G} : \mathsf{d}(\eta, \xi) \leq k\}.$$

for any $\xi \in \mathbf{G}$.

In the general theory of discretizations of Riemannian manifolds, one usually starts with the Riemannian manifold and then considers the graphs associated to it which may be said to discretize the given Riemannian manifold. The ideas are as follows: Let M be a Riemannian manifold. A subset $\mathcal{G}$ of M is said to be ϵ*–separated,* $\epsilon > 0$, if the distance between any two distinct points of $\mathcal{G}$ is greater than or equal to ϵ. If M is Riemannian complete, with Ricci curvature bounded from below, then the Bishop–Gromov theorem (see Chavel (1994, §3.4)) implies that for any ϵ–separated subset $\mathbf{G}$ of M we have

$$\operatorname{card}\{\mathcal{G} \cap B(x; r)\} \leq \text{const.}_r$$

for all $x \in M$ and $r > 0$. We define a *discretization of* M to be a graph $\mathbf{G}$ determined by an ϵ–separated subset $\mathbf{G}$ of M, for which there exists $R > 0$ such that

$$M = \bigcup_{\xi \in \mathcal{G}} B(\xi; R). \tag{1.1}$$

Then ϵ is called the *separation*, and R the *covering, radius of the discretization.* The graph structure $\mathbf{G}$ is determined by the collection of neighbors of ξ,

$$\mathsf{N}(\xi) =: \{\mathcal{G} \cap B(\xi; 3R)\} \setminus \{\xi\},$$

for each $\xi \in \mathcal{G}$. When M has Ricci curvature bounded from below, $\mathbf{G}$ has bounded geometry.

Here we work in the opposite direction, namely we start with the graph $\mathbf{G}$ possessing bounded geometry. We define the *volume of subsets of vertices* to be given by the counting measure, and we denote the volume of a finite set K in $\mathbf{G}$ by $|K|$. We shall assume that $\mathbf{G}$ has polynomial volume growth of the same order from above and below, that is, for all $\xi \in \mathbf{G}$ we have

$$\text{const.}k^{\nu} \leq |\beta(\xi; k)| \leq \text{const.}k^{\nu}, \qquad \nu \geq 1$$

for large k.

For any finite K and any subset S of K we define ∂S, *the graph boundary of S in K,* to be the subset of K consisting of all points whose distance from S is precisely equal to 1. We say that *the graph* $\mathbf{G}$ *satisfies the Neumann–Cheeger inequality* if for every $\alpha \in (0, 1)$ there exists $\text{const.}_{\alpha} > 0$ such that for every positive integer k, point $\xi \in \mathbf{G}$, and subset S of $\beta(\xi; k)$ satisfying

$$|S| \leq \alpha|\beta(\xi; k)|, \tag{1.2}$$

we have

$$\frac{|\partial S|}{|S|} \geq \frac{\text{const.}_{\alpha}}{k}. \tag{1.3}$$

Our 2–dimensional Riemannian manifold associated to $\mathbf{G}$ is constructed as follows: Let $\mathbf{M}$ denote the maximal valence of $\mathbf{G}$. To every $m = 1, \ldots, \mathbf{M}$ associate a Riemannian manifold $\mathbb{S}_m$ diffeomorphic to the 2–sphere with m holes deleted. We also require that every boundary circle of the deleted holes be collared by a Riemannian cylinder in $\mathbb{S}_m$ isometric to $\mathbb{S}^1(\epsilon) \times [0, \epsilon]$, where ϵ is independent of m (and $\mathbb{S}^1(\epsilon)$ denotes the circle of radius ϵ). To every vertex $\xi \in \mathbf{G}$ with valence $m(\xi)$ we associate the manifold with boundary, and then give a one-one correspondence between the edges emanating from ξ and the holes of $\mathbb{S}_m$. To every oriented edge in $\mathbf{G}$ we associate the Riemannian flat cylinder $\mathbb{S}^1(\epsilon) \times [-1, 1]$, and then we glue all together according to the correspondences to obtain a 2–dimensional complete Riemannian manifold M of bounded geometry. Here *bounded geometry* means that the Ricci curvature of M is bounded from below, and that the injectivity radius of M is strictly positive. If one associates to each m a particular point $x_m \in \mathbb{S}_m$, then one has a natural identification of $\mathbf{G}$ with a discretization of M.

Given any relatively compact, smoothly bounded domain in D in the 2–dimensional Riemannian manifold M, consider all compact 1–dimensional submanifolds Γ of D which divide D into two open subsets Ω_1, Ω_2. Assume that Ω_1 and Ω_2 are labelled so that $A(\Omega_1) \leq A(\Omega_2)$, and define the *Neumann–Cheeger constant* $h^N(D)$ by

$$h^N(D) = \inf_{\Gamma} \frac{\ell(\Gamma)}{A(\Omega_1)}.$$

In the above ℓ denotes arc length and A denotes area. Denote $h^N(B(o; R))$ by $h^N(o; R)$. We say that *the manifold M satisfies the Neumann–Cheeger inequality* if there exists a positive constant and $R_0 > 0$ such that

$$h^N(o;R) \geq \frac{\text{const.}}{R} \tag{1.4}$$

for all $R > R_0$ for all $o \in M$.

Of course, Cheeger's theorem then implies that M also satisfies the Neumann–Poincaré inequality (see (1996)).

Definition. By a *regular exhaustion of* M *based at* $o \in M$ we mean a sequence (E_j) of subsets of M, containing o, such that each E_j is relatively compact with smooth boundary and satisfying

(a) $E_k \subset E_{k+1}$ for all $k = 1, 2 \ldots$, and $\cup_k E_k = M$;

(b) there exists a positive constant $c_o > 1$ such that

$$B(o; {c_o}^{-1}k) \subseteq E_k \subseteq B(o; c_o k)$$

for all $k = 1, 2 \ldots$. We say that a family of regular exhaustions $(\Omega_j)_o$, $o \in M$, is *uniformly regular* if the c_o may be chosen independently of o.

Main Theorem. *If the graph* $\mathbf{G}$ *satisfies the Neumann–Cheeger inequality then there exists a uniformly regular family of exhaustions* $(\Omega_j)_o$, $o \in M$, *of the Riemannian manifold* M *and a* $\rho > 0$ *such that for every* j, *and every* $r \in (-\rho, \rho)$, *the domain* $\Omega_{j;r}$ *of* M *interior to the hypersurface of (signed) distance* r *from the boundary* $\partial\Omega_j$ *of* Ω_j *satisfies*

$$h^N(\Omega_{j;r}) \geq \frac{\text{const.}}{(j+r)^2}. \tag{1.5}$$

We would like to prove that if the graph $\mathbf{G}$ satisfies the Neumann–Cheeger inequality then the associated Riemannian manifold M also satisfies the Neumann–Cheeger inequality in the full sense defined above. However, we do not require this much for our analytic applications. What we have will suffice to drive the analytic arguments of (1996) to guarantee the validity of heat kernel bounds (0.1).

Also note that once we have the Main Theorem for the given Riemannian metric of M, the theorem remains valid for any quasi-isometric perturbation of the original metric. Note that the property of bounded geometry is not necessarily preserved; so one cannot automatically pass to the heat kernel estimates under the quasi-isometry.

The standard jungle gym in $\mathbb{R}^3$

The standard 2–dimensional jungle gym JG^2 in $\mathbb{R}^3$ is constructed by (i) considering the integer lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$, (ii) connecting points, for which precisely one of their coordinates differ by 1 and the other two coordinates are equal, by a line segment parallel to the coordinate axis, (iii) considering the surface consisting of all points with distance from the 1–dimensional network to be precisely equal to ϵ, for some given small $\epsilon > 0$, and finally (iv) smoothing out the corners in a bounded periodic fashion. To obtain a Neumann–Cheeger inequality for JG^2 one considers the following regular exhaustions $\mathcal{D}_k$ of JG^2 and $\mathcal{Z}_k$ of $\mathbb{Z}^3$: Define

$$|x|_{\mathsf{m}} = \max_{i=1,2,3} |x_i|$$

and

$$\begin{aligned} \mathcal{C}_R &= JG^2 \cap \{x \in \mathbb{R}^3 : |x|_{\mathsf{m}} < R\}, \\ \mathcal{D}_k &= \mathcal{C}_{k-1/2} \\ \mathcal{Z}_k &= \mathbb{Z}^3 \cap \mathcal{C}_{k-1/2} = \{z \in \mathbb{Z}^3 : \ |z|_{\mathsf{m}} < k\}. \end{aligned}$$

Then the argument of the Main Theorem will show that we have the Neumann–Cheeger inequality for the exhaustion $\Omega_j = \mathcal{C}_{j-1/2}$ and $r_o \in (0, 1/2 - \epsilon_o)$, for some $\epsilon_o > 0$, once we have the Neumann–Cheeger inequality for the graph-exhaustion $\mathcal{Z}_k$, $k = 1, \ldots$, that is,

Theorem 1.1. *Let S be a subset of $\mathcal{Z}_k$, satisfying*

$$|S| \leq \alpha |\mathcal{Z}_k| \tag{1.6}$$

for some $\alpha \in (0,1)$. Then

$$\frac{|\partial S|}{|S|} \geq \frac{\text{const.}_\alpha}{k}.$$

Corollary 1.1. *The jungle gym JG^2 possesses a uniformly regular family of exhaustions which satisfies the Neumann–Cheeger inequality (1.5).*

Proof. The theorem is actually Theorem 8 of Bollobás–Leader (1991), but we offer here a rather simple and elegant proof. First we note, by the Loomis–Whitney theorem (1949), that

$$|S|^2 \leq |S_1||S_2||S_3|,$$

where S_j denotes the projection of S onto the coordinate plane perpendicular to the x_j–axis, $j = 1, 2, 3$. A simple proof of the inequality ((1957, p. 161ff)) goes as follows: For every integer

z let $S_3(z)$ denote the projection onto the xy–plane of the intersection of S with the plane parallel to the xy–plane at (signed) height z. Then one easily has the 2–dimensional version:

$$|S_3(z)| \leq |(S_3(z))_1||(S_3(z))_2|,$$

where $(S_3(z))_1$ denotes the projection of $S_3(z)$ onto the x–axis, $(S_3(z))_2$ denotes the projection of $S_3(z)$ onto the y–axis. This implies

$$|S_3(z)|^2 \leq |(S_3(z))_1||(S_3(z))_2||S_3(z)| \leq |(S_3(z))_1||(S_3(z))_2||S_3|,$$

which implies

$$\begin{aligned} |S| &\leq \sum_z |S_3(z)| \\ &\leq |S_3|^{1/2} \sum_z |(S_3(z))_1|^{1/2}|(S_3(z))_2|^{1/2} \\ &\leq |S_3|^{1/2} \left\{\sum_z |(S_3(z))_1|\right\}^{1/2} \left\{\sum_z |(S_3(z))_2|\right\}^{1/2} \\ &= |S_3|^{1/2}|S_2|^{1/2}|S_1|^{1/2}, \end{aligned}$$

which is the claim.

Now that we have the Loomis–Whitney inequality, we bound the discrete Neumann–Cheeger constant from below. So $|S|$ satisfies (1.6) for some $\alpha \in (0,1)$. Let π_j denote the projection of S onto the coordinate plane perpendicular to the x_j–axis, so $S_j = \pi_j(S)$. Set

$$T_j = \{x \in S_j : \ |\pi^{-1}[x] \cap S| = 2n-1\},$$

that is, for every $x \in T_j$, the full line in Z_n over $\pi_j(x)$ is in S. Then

$$|T_j| \leq \alpha(2k-1)^2$$

for all j. Now the Loomis–Whitney inequality implies

$$|S|^2 = \alpha^2(2k-1)^{2\cdot 3} \leq |S_1||S_2||S_3|,$$

which implies there exists at least one value of j, call it i, such that

$$|S_i| \geq \alpha^{2/3}(2k-1)^2,$$

which implies

$$|S_i| - |T_i| \geq (\alpha^{2/3} - \alpha)(2k-1)^2.$$

Therefore, for each $x \in S_i \setminus T_i$ the line $\pi^{-1}[x] \cap S$ has an element of ∂S in the interior of $\mathcal{Z}_k$, which implies

$$|\partial S| \geq |S_i| - |T_i| \geq (\alpha^{2/3} - \alpha)(2k-1)^2 \geq (\alpha^{2/3} - \alpha)\frac{|S|}{2k-1},$$

which is the claim. qed

2. Proof of the main theorem

We start with some basic facts about M.

1. There exists an $\epsilon > 0$ such that every homotopically nontrivial loop has length greater than or equal to ϵ.

2. We call every holed sphere $\mathbf{S}_m$ a *joint.* To every sphere $\mathbf{S}_m$ with m handles we associate "half" the cylinder $\mathbb{S}^1(\epsilon) \times [-1, 1]$ to each of the holes, that is, we extend each of the collars of the boundary circles by the half-cylinders $\mathbb{S}^1(\epsilon) \times [-1, 0]$. Call any such subset $\mathcal{S}$ of M *an extended joint.* (Of course, in the standard jungle gym in $\mathbb{R}^n$ the subset $\mathcal{S}$ forms a fundamental domain for the discrete group $\mathbb{Z}^3$ acting on M.) There are only finitely many isometrically distinct types of (extended) joints. Therefore, there exists $\ell_0 > 0$ ($\ell_1 > 0$) such that for every (extended) joint $\mathcal{S}$, and every path in $\overline{\mathcal{S}}$ with endpoints on distinct components of $\partial\mathcal{S}$, the length of the path is greater than or equal to ℓ_0 (ℓ_1).

3. The Neumann–Cheeger inequality for the graph and the upper–lower bounds of volume growth of the graph imply (see Coulhon–Saloff-Coste ()) that the graph $\mathbf{G}$ satisfies a ν–dimensional isoperimetric inequality, that is, there exists const. > 0 such that for all finite K in $\mathbf{G}$ we have

$$\frac{|\partial K|}{|K|^{1-1/\nu}} \geq \text{const.}.$$

(For $\nu = 1$ the inequality is trivial.) This implies by a theorem of Kanai (1985) (see Chavel (1994, Chapter 6) for a detailed exposition) that M itself satisfies the isoperimetric inequality

$$\frac{\ell(\partial\Omega)}{A(\Omega)^{1-1/\nu}} \geq \text{const.},$$

for all relatively compact Ω with smooth boundary.

We now start the argument. Our uniformly regular family of exhaustions is constructed as follows: Given $o \in M$, associate to it the closed extended joint containing o. If o belongs to the boundary of an extended joint then pick one of the two extended joints. This associates to o an element ξ_o of the graph. For each positive integer k consider first the combinatorial metric disk $\beta(\xi_o; k)$ and then consider the union E_k of the extended metric joints determined by the elements of $\beta(\xi_o; k)$. We let $\mathcal{D}_k$ denote the interior of E_k. This is the desired exhaustion. We wish to show it satisfies (1.5). In what follows we shall only consider, though, the case $r = 0$. The general case is easy enough, but only considering the case $r = 0$ will make some of the exposition cleaner.

So we fix $\mathcal{D}_k$ in the 2–dimensional Riemannian manifold M, and consider all compact 1–dimensional submanifolds Γ of D which divide D into two open subsets Ω_1, Ω_2, with Ω_1 and Ω_2 labelled so that $A(\Omega_1) \leq A(\Omega_2)$. We want to show the existence of a positive constant, independent of k, so that

$$\inf_{\Gamma} \frac{\ell(\Gamma)}{A(\Omega_1)} \geq \frac{\text{const.}}{k}.$$

First, by a lemma of Yau (1975) it suffices to consider the case where Ω_1 and Ω_2 are domains.

Suppose $\Omega_1 \subset\subset \mathcal{D}_k$. Then **3.** implies

$$\text{const.} \leq \frac{\ell(\Gamma)}{A(\Omega_1)^{1-1/\nu}} = \frac{\ell(\Gamma)}{A(\Omega_1)} A(\Omega_1)^{1/\nu} \leq \frac{\ell(\Gamma)}{A(\Omega_1)} \left\{ \frac{A(\mathcal{D}_k)}{2} \right\}^{1/\nu} \leq \text{const.} k \frac{\ell(\Gamma)}{A(\Omega_1)},$$

which implies

$$\frac{\ell(\Gamma)}{A(\Omega_1)} \geq \frac{\text{const.}}{k}.$$

So this case is easily covered.

Next, given any Γ with associated domain Ω_1, consider any extended joint $\mathcal{S}$ in M for which

$$\ell(\Gamma \cap \overline{\mathcal{S}}) < \epsilon_1 =: \frac{1}{2} \min \{\epsilon, \ell_1\}. \tag{2.1}$$

(ϵ was defined in **1.** and ℓ in **2.**) Then $\Gamma \cap \mathcal{S}$ is either a union of closed curves, in which case it cannot be homotopically nontrivial. Therefore, either $\Gamma \cap \mathcal{S} = \Gamma$ and $\Omega_1 \subset\subset \mathcal{S}$, which we just discussed. Or $\Gamma \cap \mathcal{S}$ is a union of curves with endpoints located on the boundary $\partial \mathcal{S}$ of $\mathcal{S}$. But then Γ cannot connect distinct components of $\partial \mathcal{S}$ (it will be too long!); therefore each of the components of $\Gamma \cap \mathcal{S}$ is homotopically trivial relative to some component of $\partial \mathcal{S}$.

Now assume

$$\ell(\Gamma \cap \overline{\mathcal{S}}) < \epsilon_1, \qquad A(\Omega_1 \cap \mathcal{S}) \leq A(\mathcal{S})/2. \tag{2.2}$$

For convenience, assume $\Gamma_1 \cap \mathcal{S}$ has only one boundary component. Then $\Omega_1 \cap \mathcal{S}$ is bounded by $\Gamma \cap \mathcal{S}$ and Γ_1 the shortest part of the circle in $\partial\mathcal{S}$ connecting the endpoints of $\Gamma \cap \mathcal{S}$. Let S be the extended joint neighboring $\mathcal{S}$ whose boundary intersects Γ_1. If S also satisfies the estimate (2.2) then $\Omega_1 \subset\subset \mathcal{D}_k$ and we already have the lower bound. Therefore consider the case where S does not satisfy (2.2); and set (see Figure 1)

$$\begin{aligned} D' &= \overline{\mathcal{S}} \cap \Omega_1, & \qquad D'' &= \overline{\mathsf{S}} \cap \Omega_1, \\ \Gamma' &= \mathcal{S} \cap \Gamma, & \qquad \Gamma'' &= \mathsf{S} \cap \Gamma. \end{aligned}$$

Now the original (Dido) isoperimetric inequality implies

$$\ell(\Gamma') \geq \text{const.}\sqrt{A(D')}.$$

Because the area is bounded from above, we have

$$\ell(\Gamma') \geq \text{const.}A(D'),$$

which implies

$$\begin{aligned} \frac{\ell(\Gamma') + \ell(\Gamma'')}{A(D') + A(D'')} &= \frac{(1/2)\{\ell(\Gamma') + \ell(\Gamma')\} + \ell(\Gamma'')}{A(D') + A(D'')} \\ &\geq \frac{(1/2)\text{const.}A(D') + (1/2)\ell(\Gamma_1) + \ell(\Gamma'')}{A(D') + A(D'')} \\ &\geq \text{const.}\frac{A(D') + \ell(\Gamma_1) + \ell(\Gamma'')}{A(D') + A(D'')} \\ &\geq \text{const.}\frac{\ell(\Gamma_1) + \ell(\Gamma'')}{A(D'')}. \end{aligned}$$

So it suffices to consider those cases where (2.2) is not valid. Note that if $\mathcal{S}$ had no *neighboring* extended joint in the sense described above, then $\mathcal{S}$ is a boundary extended joint of $\mathcal{D}_k$, in which case the estimate follows from the original (Dido) isoperimetric inequality.

So are now assuming that

$$\ell(\Gamma \cap \overline{\mathcal{S}}) < \epsilon_1, \qquad A(\Omega_1 \cap \mathcal{S}) > A(\mathcal{S})/2. \tag{2.3}$$

Then enlarge $\Omega_1 \cap \mathcal{S}$ by replacing $\Gamma \cap \mathcal{S}$ with Γ_1. Denote the new domain by $(\Omega^*)_1$ with boundary Γ^*. Then $\ell(\Gamma^*) \leq \ell(\Gamma)$ and $A((D^*)') \geq A(D')$. The new domain will have the property that the component of $\Omega^* \cap \partial\mathcal{S}$ which intersects (the original) Γ_1 will consist of this complete bounding circle.

It therefore suffices to study those Γ for which

$$\ell(\Gamma \cap \overline{\mathcal{S}}) \geq \epsilon_1$$

for all $\mathcal{S}$ for which $\Gamma \cap \overline{\mathcal{S}} \neq \emptyset$. Set

$$\mathfrak{S} = \{\mathcal{S} : \ \Omega_1 \cap \mathcal{S} \neq \emptyset\},$$

and for any fixed $\delta > 0$, pick

$$\alpha_\delta = \frac{\epsilon_1}{(1-\delta)A(\mathcal{S})}, \qquad \beta_\delta = \min\{\alpha_\delta, 1\}$$

and partition $\mathfrak{S}$ into $\mathfrak{S}_1$, $\mathfrak{S}_2$:

$$\mathfrak{S}_1 = \{\mathcal{S} \in \mathfrak{S} : A(\Omega_1 \cap \mathcal{S}) \leq (1-\delta)A(\mathcal{S})\}, \qquad \mathfrak{S}_2 = \{\mathcal{S} \in \mathfrak{S} : A(\Omega_1 \cap \mathcal{S}) > (1-\delta)A(\mathcal{S})\}.$$

Then

$$\ell(\Gamma \cap \overline{\mathcal{S}}) \geq \alpha_\delta A(\Omega_1 \cap \mathcal{S})$$

for all $\mathcal{S} \in \mathfrak{S}_1$, which implies

$$\begin{aligned}
\frac{\ell(\Gamma)}{A(\Omega_1)} &\geq \frac{1}{2}\frac{\sum_{\mathfrak{S}_1} \ell(\Gamma \cap \overline{\mathcal{S}}) + \sum_{\mathfrak{S}_2} \ell(\Gamma \cap \overline{\mathcal{S}})}{\sum_{\mathfrak{S}_1} A(\Omega_1 \cap \mathcal{S}) + \sum_{\mathfrak{S}_2} A(\Omega_1 \cap \mathcal{S})} \\
&\geq \frac{1}{2}\frac{\beta_\delta \sum_{\mathfrak{S}_1} A(\Omega_1 \cap \mathcal{S}) + \sum_{\mathfrak{S}_2} \ell(\Gamma \cap \overline{\mathcal{S}})}{\sum_{\mathfrak{S}_1} A(\Omega_1 \cap \mathcal{S}) + \sum_{\mathfrak{S}_2} A(\Omega_1 \cap \mathcal{S})} \\
&\geq \frac{\alpha_\delta}{2}\frac{\sum_{\mathfrak{S}_1} A(\Omega_1 \cap \mathcal{S}) + \sum_{\mathfrak{S}_2} \ell(\Gamma \cap \overline{\mathcal{S}})}{\sum_{\mathfrak{S}_1} A(\Omega_1 \cap \mathcal{S}) + \sum_{\mathfrak{S}_2} A(\Omega_1 \cap \mathcal{S})} \\
&\geq \frac{\beta_\delta}{2}\frac{\sum_{\mathfrak{S}_2} \ell(\Gamma \cap \overline{\mathcal{S}})}{\sum_{\mathfrak{S}_2} A(\Omega_1 \cap \mathcal{S})} \\
&\geq \frac{\beta_\delta}{2}\frac{\epsilon_1 \text{card}\{\mathcal{S} \in \mathfrak{S}_2 : \ \Gamma \cap \overline{\mathcal{S}} \neq \emptyset\}}{\sum_{\mathfrak{S}_2} A(\Omega_1 \cap \mathcal{S})}.
\end{aligned}$$

To bound this last quotient from below, set

$$\mathfrak{S}_2{}' = \{\mathcal{S} \in \mathfrak{S}_2 : \ \Gamma \cap \overline{\mathcal{S}} \neq \emptyset\},$$

and note that

$$\frac{A(\mathcal{D}_k)}{2} \geq A(\Omega_1) \geq \sum_{\mathfrak{S}_2} A(\Omega_1 \cap \mathcal{S}) \geq (1-\delta)A(\mathcal{S})\mathrm{card}\,{\mathfrak{S}_2}',$$

which implies

$$|{\mathfrak{S}_2}'| \leq \frac{(1-\delta)^{-1}}{2}|\mathcal{B}(\xi_o;k)|. \tag{2.4}$$

Let S be the subset of $\mathbf{G}$ corresponding to $\mathcal{S} \in {\mathfrak{S}_2}'$, and ∂S the graph boundary of S — that is, ∂S consists of all points in $\mathbf{G}$ whose graph-distance from S is precisely equal to 1. But since any point in ∂S has graph distance 1 from at most $\mathbf{M}$ elements of S, we have

$$|\partial S| \leq \frac{1}{\mathbf{M}}|{\mathfrak{S}_2}'|.$$

Then we have

$$\frac{\ell(\Gamma)}{A(\Omega_1)} \geq \frac{\beta_\delta}{2}\frac{\epsilon_1|{\mathfrak{S}_2}'|}{\sum_{\mathfrak{S}_2} A(\Omega_1 \cap \mathcal{S})} \geq \frac{\beta_\delta\epsilon_1\mathrm{const.}}{2A(\mathcal{S})}\frac{|\partial S|}{|S|},$$

which implies the Main Theorem. qed

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