IV. The Fourier Transform

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Contents

§1. Definition of the Fourier transform

We think of Fourier series as a discrete Fourier transform, namely, given a piecewise continuous $2\pi-$periodic function ϕ, we associate with it the sequence of Fourier coefficients (here viewed as a function on the integers) given by

$$\widehat{\phi}(n) = \sqrt{2\pi}c_n(\phi) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \phi(x)e^{-inx}\,dx.$$

(This is slightly different from the way we wrote the transform in Remark III.2.) From knowledge of $\widehat{\phi}(n)$ one recovers the original function $\phi(x)$ by

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{+\infty} \widehat{\phi}(n)e^{inx}.$$

So what we call here the “transform” is the calculation of the Fourier coefficients of ϕ. The “inverse transform” is the Fourier series itself.

The integral Fourier transform goes as follows: Given a function $f(x)$, $-\infty < x < +\infty$, one associates with it the **Fourier transform of** f, $\widehat{f}(\xi)$, given by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi}\, dx. \tag{1}$$

The candidate for recovering f from knowledge of $\widehat{f}$ is, then,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{f}(\xi) e^{ix\xi}\, d\xi. \tag{2}$$

One important difficulty with the definition of $\widehat{f}$ is that $f(x)$ may be as smooth as one wishes, and yet the integral $\int_{-\infty}^{+\infty} f(x)e^{-ix\xi}\, dx$ might not even exist. Simply try the function $f(x) = 1$ for all x. So we, naturally, have to assume something about f, namely, that there exists a constant $M > 0$ so that for any $A, B > 0$ we have

$$\int_{-A}^{B} |f(x)|\, dx < M.$$

Then one has well-defined

$$\int_{-\infty}^{+\infty} |f(x)|\, dx \qquad \text{and} \qquad \int_{-\infty}^{+\infty} f(x)\, dx.$$

We call the collection of such functions L^1, and it is common to write

$$||f||_1 = \int_{-\infty}^{+\infty} |f(x)|\, dx.$$

We mention some facts about L^1 to be used in what follows:

A. If $[a, b]$ is an interval in $(-\infty, +\infty)$, and $I_{[a,b]}$ is the *indicator function* of $[a, b]$, that is,

$$I_{[a,b]} = \begin{cases} 1 & a \le x \le b \\ 0 & \text{otherwise} \end{cases},$$

then $I_{[a,b]}$ is certainly an element of L^1. We refer to any finite linear combination of such indicator functions as a *step function*. Certainly, any step function is an element of L^1. Moreover, it is a fact that given any f in L^1, one can find a sequence of step functions ϕ_n for which

$$\lim_{n\to\infty} ||f - \phi_n||_1 = 0.$$

B. Recall from §III.2 that we say a function f *has bounded support* if there exists an interval $[a, b]$ so that f is identically equal to 0 off that interval. Step functions have bounded support. It is a fact that given any function $f(x)$ in L^1, then for any $k = 0, 1, \ldots, \infty$, one can find a sequence of boundedly supported functions ψ_n in C^k, for which

$$\lim_{n\to\infty} ||f - \psi_n||_1 = 0.$$

(Recall that C^k, $k \geq 1$, denotes the set of functions which have k continuous derivatives. When $k = 0$, then C^0 denotes the collection of continuous functions.)

C. The **Lebesgue dominated convergence theorem** states that if we are given functions f, g in L^1, with $g \geq 0$, and a sequence of functions f_n in L^1 for which

$$|f_n(x)| \leq g(x), \qquad \lim_{n\to\infty} f_n(x) = f(x)$$

for all x (except for, at most, our usual exceptional set), then

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f_n(x)\, dx = \int_{-\infty}^{+\infty} f(x)\, dx.$$

Exercises

Exercise 1. Calculate the Fourier transform of

$$f(x) = \begin{cases} 1 & |x - a| < R \\ 0 & |x - a| > R \end{cases}.$$

Exercise 2. Calculate the Fourier transform of

$$f(x) = \begin{cases} 1 - |x|/R & |x| < R \\ 0 & |x| > R \end{cases}.$$

Exercise 3. Calculate the Fourier transform of

$$f(x) = e^{-|x|}.$$

§2. Basic properties

For piecewise continuous functions f, with finite $||f||_1$, we therefore have $\widehat{f}$ well-defined by (1). Certainly we have

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, dx, \qquad \text{and} \qquad |\widehat{f}(\xi)| \leq ||f||_1/\sqrt{2\pi} \tag{3}$$

for all ξ. So $\widehat{f}$ is always bounded. One easily verifies

Proposition 1. *For fixed y in $\mathbb{R}$ we have*

$$g(x) = f(x-y) \qquad \Rightarrow \qquad \widehat{g}(\xi) = e^{-iy\xi}\widehat{f}(\xi), \tag{4}$$

and

$$h(x) = e^{iyx}f(x) \qquad \Rightarrow \qquad \widehat{h}(\xi) = \widehat{f}(\xi - y). \tag{5}$$

Proposition 2. *For given λ in $\mathbb{R}$ we have*

$$K_\lambda(x) = \frac{1}{\lambda} f\left(\frac{x}{\lambda}\right) \qquad \Rightarrow \qquad \widehat{K_\lambda}(\xi) = \widehat{f}(\lambda\xi), \tag{6}$$

and

$$H_\lambda(x) = f(\lambda x) \qquad \Rightarrow \qquad \widehat{H}_\lambda(\xi) = \frac{1}{\lambda}\widehat{f}\left(\frac{\xi}{\lambda}\right). \tag{7}$$

Definition. For functions f, g with finite $||f||_1$, $||g||_1$, define their *convolution* $f * g$ by

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-y)g(y)\, dy.$$

One then has

Proposition 3. *The convolution $f * g$ is defined for almost all x, and*

$$(f*g)\widehat{\ }(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{8}$$

Theorem 1. *We have*

$$\phi(x) = e^{-x^2/2} \qquad \Rightarrow \qquad \widehat{\phi}(\xi) = e^{-\xi^2/2}.$$

Proof. We give two arguments. The straightforward one goes as follows:

$$\begin{aligned}
\widehat{\phi}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2} e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(1/2)\{x^2+2ix\xi\}}\, dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \int_{-\infty}^{+\infty} e^{-\{x+i\xi\}^2/2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx \\
&= e^{-\xi^2/2}.
\end{aligned}$$

The next to last equality is obtained by an argument using Cauchy's integral theorem (from Complex Function Theory). qed

We give a second, more slick, argument below.

Theorem 2. The Riemann–Lebesgue Lemma. *For* $||f||_1$ *finite we have*

$$\lim_{\xi\to\infty} \widehat{f}(\xi) = 0.$$

Proof. First, consider the case when f is the indicator function of some interval $[a,b]$, that is, $f = I_{[a,b]}$. Then

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-ib\xi} - e^{-ia\xi}}{i\xi},$$

which implies

$$|\widehat{f}|(\xi) \le \frac{1}{\sqrt{2\pi}} \frac{2}{|\xi|} \to 0$$

as $\xi \to \infty$. So the theorem is valid for indicator functions. For a step function ϕ we have the representation

$$\phi = \sum_{j=1}^{N} \alpha_j I_{[a_j,b_j]},$$

which implies

$$\widehat{\phi} = \sum_{j=1}^{N} \alpha_j \widehat{I_{[a_j,b_j]}},$$

from which one has

$$|\widehat{\phi}|(\xi) \leq \frac{\max_j |\alpha_j|}{\sqrt{2\pi}} \frac{2N}{|\xi|} \to 0.$$

So the lemma is valid for step functions. For arbitrary f, approximate f by the step function ϕ. Then (4) implies

$$|\widehat{f}(\xi)| \leq |\widehat{f}(\xi) - \widehat{\phi}(\xi)| - |\widehat{\phi}(\xi)| \leq \frac{1}{\sqrt{2\pi}} ||f-\phi||_1 + |\widehat{\phi}(\xi)|.$$

Now $\widehat{\phi} \to 0$ as $\xi \to \infty$. So for large ξ, $|\widehat{f}|$ is essentially estimated by const.$||f-\phi||_1$, which can be made as small as we like, by **A.** above. qed

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be an element of L^1, continuous on all of $\mathbb{R}$, and have piecewise continuous f' in L^1. Then*

$$\{f'\}\widehat{\ }(\xi) = (i\xi)\widehat{f}(\xi)$$

Proof. We use integration-by-parts:

$$\int_{-R}^{R} f'(x)e^{-ix\xi}\,dx = f(x)e^{-ix\xi}\Big|_{-R}^{R} + i\xi \int_{-R}^{R} f(x)e^{-ix\xi}\,dx.$$

We want to show that

$$\lim_{x\to\infty} f(x) = 0;$$

for the theorem will then follow immediately. Well,

$$f(x) = f(0) + \int_0^x f'(t)\,dt,$$

and f' is an element of L^1. So the integral on the right hand side has a limit $f(+\infty)$ as $x \to +\infty$, and a limit $f(-\infty)$ as $x \to -\infty$. But, if the limits exist and are not both equal to 0, then the integral $\int |f(x)|\,dx$ could not be finite. We conclude that both:

$$f(+\infty) = f(-\infty) = 0,$$

and the theorem follows. qed

Corollary 1. *If for $k \geq 1$, f has continuous derivatives up to order $k-1$, all in L^1 , and the k^{th} derivative $f^{(k)}$ is piecewise continuous and in L^1, then*

$$\widehat{f^{(k)}}(\xi) = (i\xi)^k \widehat{f}(\xi). \tag{9}$$

In particular, the Riemann–Lebesgue lemma implies

$$\lim_{\xi \to \infty} \xi^k \widehat{f}(\xi) = 0. \tag{10}$$

What about the differentiability of $\widehat{f}(\xi)$ with respect to ξ? Well,

$$\frac{\widehat{f}(\xi+\eta) - \widehat{f}(\xi)}{\eta} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} \left\{ \frac{e^{-ix\eta} - 1}{\eta} \right\} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} -ixf(x) e^{-ix\xi} \left\{ \frac{e^{-ix\eta} - 1}{-ix\eta} \right\} dx.$$

Since the bracket in the last integral is, at best, bounded we will certainly require the convergence of

$$\int_{-\infty}^{+\infty} |xf(x)|\, dx$$

that is, f must tend to 0 sufficiently quickly to guarantee the existence of $d\widehat{f}/d\xi$. When this is so, the Lebesgue dominated convergence implies we may pass to the limit (as $\eta \to 0$) under the integral sign to obtain

$$\{\widehat{f}\}'(\xi) = (-ixf)\widehat{\ }(\xi).$$

More generally, with the same argument, we have

Theorem 4. *If for $k \geq 1$*

$$\int_{-\infty}^{+\infty} |x^k f(x)|\, dx < +\infty$$

then $\widehat{f}$ has derivatives up to order k, and

$$\widehat{f}^{(j)}(\xi) = \{(-ix)^j f\}\widehat{\ }(\xi), \tag{11}$$

for $j = 1, \ldots, k$.

Second proof of Theorem 1. Note that ϕ given by

$$\phi(x) = e^{-x^2/2}$$

satisfies the differential equation

$$\phi' + x\phi = 0.$$

Subject the equation to the Fourier transform. Then

$$i\xi\widehat{\phi} + i\widehat{\phi}' = 0,$$

that is, $\widehat{\phi}$ satisfies the same differential equation. Therefore

$$\widehat{\phi}(\xi) = ce^{-\xi^2/2}, \qquad c = \widehat{\phi}(0) = 1.$$

This implies the claim. qed

§3. The inversion and Plancherel formulae

Definition. We let $\mathcal{S}$ denote the collection of *Schwartz functions*, that is, functions defined on $\mathbb{R}$ which are continuous C^∞ for which: given any integers $N, k > 0$, then

$$\lim_{x\to\infty} |x|^N |f^{(k)}(x)| = 0.$$

Then Corollary 1, the Riemann–Lebesgue Lemma, and Theorem 4, imply that if the function f is in $\mathcal{S}$, then its Fourier transform $\widehat{f}$ is also in $\mathcal{S}$.

Example 1. One easily sees that the function $\phi(x) = e^{-x^2/2}$ is an element of $\mathcal{S}$.

Theorem 5. *If the function f is an element of L^1, and ϕ is an element of $\mathcal{S}$ then*

$$\int_{-\infty}^{+\infty} \phi(\xi)\widehat{f}(\xi)e^{ix\xi}\,d\xi = \int_{-\infty}^{+\infty} f(y)\widehat{\phi}(y-x)\,dy. \tag{12}$$

Proof. One derives (12) by starting with the right hand side, writing out $\widehat{\phi}(y-x)$, and changing the order of integration. qed

Theorem 6. Fourier inversion formula. *If f is an element of $\mathcal{S}$ then f can be recovered from knowledge of $\widehat{f}$, by (2), that is*

$$f(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{+\infty} \widehat{f}(\xi) e^{ix\xi}\, d\xi. \tag{13}$$

Proof. We want to apply (12) to a collection of ϕ_λ so that $\widehat{\phi_\lambda}$ will form an approximate identity. It goes as follows: Given $\psi(x)$ in $\mathcal{S}$ with $\widehat{\psi}(\xi) \geq 0$ for all ξ and with

$$\int_{-\infty}^{+\infty} \widehat{\psi}(\xi)\, d\xi = 1,$$

we want, in (12), for each λ,

$$\widehat{\phi_\lambda}(y - x) = \frac{1}{\lambda} \widehat{\psi}\left(\frac{y - x}{\lambda}\right).$$

So we set

$$\phi_\lambda(\xi) = \psi(\lambda\xi)$$

by (6). Then, as $\lambda \downarrow 0$, the left hand side of (12)

$$\int_{-\infty}^{+\infty} \phi_\lambda(\xi)\widehat{f}(\xi)e^{ix\xi}\, d\xi \to \psi(0) \int_{-\infty}^{+\infty} \widehat{f}(\xi)e^{ix\xi}\, d\xi,$$

and the right hand side will tend to

$$\int_{-\infty}^{+\infty} f(y)\widehat{\phi_\lambda}(y - x)\, dy \to f(x);$$

so

$$f(x) = \psi(0) \int_{-\infty}^{+\infty} \widehat{f}(\xi)e^{ix\xi}\, d\xi.$$

It therefore remains to pick $\psi(x)$. Well, pick

$$\psi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and use Theorem 1. This will then imply (13). qed

Theorem 7. Plancherel formula. *If f is in $\mathcal{S}$, then*

$$\int_{-\infty}^{+\infty} |\widehat{f}(\xi)|^2\, d\xi = \int_{-\infty}^{+\infty} |f(x)|^2\, dx.$$

Proof. In (12) we pick

$$\phi(\xi) = \overline{\widehat{f}}(\xi) e^{-ix\xi};$$

then the left hand side of (12) becomes $\int |\widehat{f}|^2$. So we want to show

$$\widehat{\phi}(y - x) = \overline{f}(y).$$

But that follows from (5). qed

Theorem 8. Shannon sampling theorem. *Suppose $f(x) = 0$ for all $|x| \geq T > 0$. Then $\widehat{f}(\xi)$ is completely determined by its values at $\{\xi = \pi n/T : n = \text{integer}\}$.*

Proof. Let $G(\xi)$ be the $2T$–periodic function on $(-\infty, +\infty)$ for which $G = f$ on $[-T, T]$. Expand G in the Fourier series

$$G(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i\pi n x/T};$$

then

$$c_n = \frac{1}{2T} \int_{-T}^{T} G(x) e^{-i\pi n x/T}\, dx = \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\pi n x/T}\, dx = \frac{\sqrt{2\pi}}{2T} \widehat{f}(\pi n/T)$$

Therefore, for x in $[-T, T]$, $f(x) = G(x)$ is completely determined by the values of $\widehat{f}$ at the lattice $\{\xi = n\pi/T : n = \text{integer}\}$. Thus f is completely determined by these values of $\widehat{f}$, then the Fourier transform implies that so is $\widehat{f}$ itself. The precise calculation is:

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} f(x) e^{-ix\xi}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} G(x) e^{-ix\xi}\, dx \\
&= \frac{1}{2} \sum_n \widehat{f}(\pi n/T) \int_{-T}^{T} e^{i(\pi n/T - \xi)x}\, dx \\
&= \sum_n \widehat{f}(\pi n/T) \frac{\sin\{\xi T - \pi n\}}{\xi T - \pi n}. \text{ qed}
\end{aligned}$$

Remark 1. If one starts with $\widehat{f}$ with only bounded support, then one uses the Fourier inversion formula to determine f by its values on the appropriate discrete set of points.

Exercises

Exercise 4. Consider the initial value problem for the heat equation on the real line, namely we wish to solve, for $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad |u(\ ,t)| = \text{ bounded for every } t, \qquad \lim_{t\downarrow 0} u(x,t) = \phi(x) \text{ for all } x,$$

where $\phi(x)$ is continuous and bounded.

(a) Use the Fourier transform to derive a solution of the form

$$u(x,t) = \int_{-\infty}^{+\infty} (4\pi t)^{-1/2} e^{-(x-\xi)^2/4t} \phi(\xi)\, d\xi. \tag{14}$$

HINT: To derive a candidate solution, assume that for every $t > 0$, the function $x \mapsto u(x,t)$ is a Schwartz function. Take the Fourier transform of the heat equation with respect to the space variable. Obtain an ordinary differential equation for $\widehat{u(\xi,t)}$ in t (with ξ as parameter); solve it; and then use the inverse transform to obtain $u(x,t)$ given by (14).

(b) Check that $u(x,t)$ given by (14) is in fact a solution of the heat equation.

(c) Check that $u(x,t)$ given by (14) satisfies

$$\lim_{t\downarrow 0} u(x,t) = \phi(x) \text{ for all } x,$$

that is, for **heat kernel** , or **fundamental solution of the heat equation**:

$$\mathcal{E}(x,t;\xi) := (4\pi t)^{-1/2} e^{-(x-\xi)^2/4t},$$

one has

$$\lim_{t\downarrow 0} \mathcal{E}(x,t;\xi) = \delta_\xi(x).$$

HINT: See Example II.3.

(d) Show that

$$|u(x,t)| \le \sup_x |\phi(x)|.$$

(e) Show that

$$\int_{-\infty}^{+\infty} u(x,t)\, dx = \int_{-\infty}^{+\infty} \phi(x)\, dx.$$

Exercise 5. The L–periodic heat kernel is given by (see §II.2)

$$\mathbf{E}(x,t:\xi) = \sum_{k=-\infty}^{+\infty} \mathcal{E}(x,t;\xi + kL).$$

Show that, on the other hand, the methods of separation of variables and Fourier series (see Exercise III.11) imply that

$$\mathbf{E}(x,t;\xi) = \sum_{n=-\infty}^{+\infty} \frac{1}{L} e^{-(4\pi^2 n^2/L^2)t} e^{i(2\pi n/L)(x-\xi)};$$

Therefore, we have a special case of the **Poisson summation formula**:

$$\frac{1}{L}\sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 t/L^2} e^{2\pi i n x/L} = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{+\infty} e^{-(x+kL)^2/4t}.$$

For $x = 0$ we have the celebrated **Jacobi identity for theta functions**:

$$\frac{1}{L}\sum_{n=-\infty}^{+\infty} e^{-4\pi^2 n^2 t/L^2} = (4\pi t)^{-1/2} \sum_{k=-\infty}^{+\infty} e^{-k^2 L^2/4t}$$

Exercise 6. Poissson summation formula. Let ϕ be an element of $\mathcal{S}$ on $(-\infty,+\infty)$, and consider the function

$$F(x) = \sum_{n=-\infty}^{+\infty} \phi(x + 2\pi n).$$

Check that F is well-defined and periodic, with period equal to 2π. Expand F in a Fourier series

$$F(x) = \sum_{n=-\infty}^{+\infty} c_n(F) e^{inx},$$

and use the Fourier expansion of F to show that

$$\frac{1}{\sqrt{2\pi}} \sum_n \widehat{\phi}(n) = \sum_k \phi(2\pi k).$$

See what results when

(a) $\phi(x) = e^{-x^2 t/4\pi^2}$.

(b) $\phi(x) = e^{-|x|}$. Note. Even though ϕ in this example is not an element of $\mathcal{S}$ it decreases sufficiently quickly, as $x \to \infty$, to guarantee the validity of the summation formula in this case.

Exercise 7. Dirichlet problem in the Euclidean upper half-plane. We are given a function $\phi(x)$, x in $(-\infty,+\infty)$, and we seek a function $u(x,y)$, $y > 0$, for which

$$\Delta u = 0$$

on $\{(x,y) : y > 0\}$, and with boundary values

$$\lim_{y\downarrow 0} u(x,y) = \phi(x).$$

Use Fourier transforms to derive the formula

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{y}{(\xi - x)^2 + y^2}\phi(\xi)\,d\xi.$$

Verify that the formula is, in fact a solution to the problem.

§4. The Fourier transform in higher dimensional space

Here we start with a function $f : \mathbb{R}^n \to \mathbb{C}$ and define its n–**dimensional Fourier transform**, $\widehat{f}$, by

$$\widehat{f}(\boldsymbol{\xi}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(\mathbf{x})e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\,d\mathbf{x},$$

where now

$$\mathbf{x} := (x_1, \dots, x_n), \qquad \boldsymbol{\xi} := (\xi, \dots, \xi_n),$$

and

$$d\mathbf{x} \;=\; dx_1 \cdots dx_n$$

is the usual volume element of $\mathbb{R}^n$. The Fourier transform $\widehat{f}$ is well-defined for functions with finite

$$||f||_1 = \int_{\mathbb{R}^n} |f(\mathbf{x})|\,d\mathbf{x}.$$

The earlier properties on $\mathbb{R}^1$ now become

$$|\widehat{f}(\boldsymbol{\xi})| \le \frac{||f||_1}{(2\pi)^{n/2}}; \qquad g(\mathbf{x}) = f(\mathbf{x}-\mathbf{y}) \;\Rightarrow\; \widehat{g}(\boldsymbol{\xi}) = e^{-i\mathbf{y}\cdot\boldsymbol{\xi}}\widehat{f}(\boldsymbol{\xi}); \qquad \{e^{i\mathbf{y}\cdot\mathbf{x}}f\}\widehat{\ }(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}-\mathbf{y});$$

also,

$$\{f(\lambda\mathbf{x})\}\widehat{\ }(\boldsymbol{\xi}) = \lambda^{-n}\widehat{f}(\boldsymbol{\xi}/\lambda); \qquad \{f * g\}\widehat{\ }(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi})\widehat{g}(\boldsymbol{\xi}); \qquad \lim_{|\boldsymbol{\xi}|\to+\infty} \widehat{f}(\boldsymbol{\xi}) = 0.$$

and

$$\phi(\mathbf{x}) = e^{-|\mathbf{x}|^2/2} \quad\Rightarrow\quad \widehat{\phi}(\boldsymbol{\xi}) = e^{-|\boldsymbol{\xi}|^2/2}.$$

We again consider the collection of functions on $\mathbb{R}^n$, $\mathcal{S}$, which, with all their partial derivatives of all orders, have rapid decrease on $\mathbb{R}^n$. For functions f in $\mathcal{S}$ we have

$$\{\partial f/\partial x_j\}\widehat{\ }(\boldsymbol{\xi}) = i\xi_j\widehat{f}(\boldsymbol{\xi}), \qquad (\partial\widehat{f}/\partial\xi_j)(\boldsymbol{\xi}) = \{-ix_jf\}\widehat{\ }(\boldsymbol{\xi}).$$

Again the Riemann–Lebesgue lemma implies, with these last two formulae, that the Fourier transforms maps $\mathcal{S}$ into $\mathcal{S}$. Also, we easily have the higher–dimensional analog of (12).

One easily generalizes the discussion of delta functions, presented in Chapter III, to higher dimensions, and uses the same argument to obtain the **Fourier inversion formula:**

$$f(\mathbf{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}.$$

Exercises

Exercise 8. Heat diffusion on Euclidean space. Use Fourier transforms to derive the free space heat kernel for the heat equation on $\mathbb{R}^n$. In particular, show that

$$\mathcal{E}(\mathbf{x}, \boldsymbol{\xi}, t) = (4\pi t)^{-n/2} e^{-|\mathbf{x}-\boldsymbol{\xi}|^2/4t}$$

is a heat kernel on $\mathbb{R}^n$, that is, for any $\phi(\mathbf{x})$ bounded and continuous on $\mathbb{R}^n$, the function

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} \mathcal{E}(\mathbf{x}, \boldsymbol{\xi}, t)\phi(\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

is a solution to the heat equation

$$\Delta u(\mathbf{x}, t) = \frac{\partial u}{\partial t}(\mathbf{x}, t),$$

satisfying

$$u(\mathbf{x}, 0) = \phi(\mathbf{x}).$$

Exercise 9. Heat diffusion to steady state on the Euclidean upper half-plane (I), $\mathbb{R}^2_+$. The Euclidean upper half-plane $\mathbb{R}^2_+$ consists of points (x, y) in $\mathbb{R}^2$ with $y > 0$. Solve the initial-boundary value problem for the heat equation

$$\Delta u = \frac{\partial u}{\partial t}$$

on the upper half-plane, where

$$u((x, y), 0) = \phi(x, y), \qquad u(((x, 0), t) = 0,$$

where $\phi(x, y)$ is continuous, with bounded support on $\mathbb{R}^2_+$. Hint: The argument adapts “the method of images” from classical potential theory. Namely, for vanishing boundary values on $\mathbb{R} \times \{0\}$, consider the **Dirichlet heat kernel** :

$$Q((x, y), (\xi, \eta), t) = \mathcal{E}((x, y), (\xi, \eta), t) - \mathcal{E}((x, y), (\xi, -\eta), t), \qquad \text{for all } \ x, \xi \text{ in } \mathbb{R}, \ y, \eta > 0, \ t > 0.$$

Then a natural candidate for solving the initial-boundary value problem, above, is

$$u((x,y),t) = \iint_{\mathbb{R}^2_+} Q((x,y),(\xi,\eta),t)\phi(\xi,\eta)\,d\xi d\eta.$$

Check that this proposed solution is in fact a solution. What happens when $t \uparrow +\infty$?

Exercise 10. Heat diffusion to steady state on the Euclidean upper half-plane (II). Solve the initial-boundary value problem for the heat equation

$$\Delta u = \frac{\partial u}{\partial t}$$

on the upper half-plane, where

$$u((x,y),0) = 0, \qquad u(((x,0),t) = \psi(x),$$

where $\psi(x)$ is continuous, with bounded support on $\mathbb{R}$. Show that

$$u((x,y),t) = \int_0^t ds \int_{-\infty}^{+\infty} \frac{y}{s} e^{-\{(x-\xi)^2+y^2\}/4s}\phi(\xi)\,d\xi.$$

Investigate what happens when $t \uparrow +\infty$ by switching the order of integration, and using the substitution in the ds integral:

$$\rho = \frac{(x-\xi)^2+y^2}{4s}.$$

§5. CT Scan

If we are given a function $f(\mathbf{x})$, $\mathbf{x}$ in $\mathbb{R}^2$, on the plane, and we only know the integrals of the function over all lines in the plane, can we reconstruct the function? The answer is: yes. The argument goes as follows: Start with

$$\widehat{f}(\boldsymbol{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\,d\mathbf{x},$$

and introduce polar coordinates in the *transform variable.* So we let

$$\boldsymbol{\xi} = r\boldsymbol{\omega}, \qquad \text{where} \qquad r>0, \quad \boldsymbol{\omega} \text{ in } \mathbb{S}^1.$$

Then

$$\begin{aligned}
\widehat{f}(r\boldsymbol{\omega}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{x})e^{-ir\mathbf{x}\cdot\boldsymbol{\omega}}\,d\mathbf{x} \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\sigma \int_{\mathbf{x}\cdot\boldsymbol{\omega}=\sigma} f(x)e^{-ir\mathbf{x}\cdot\boldsymbol{\omega}}\,ds_{\boldsymbol{\omega},\sigma}(\mathbf{x}) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ir\sigma}\,d\sigma \int_{\mathbf{x}\cdot\boldsymbol{\omega}=\sigma} f(\mathbf{x})\,ds_{\boldsymbol{\omega},\sigma}(\mathbf{x}).
\end{aligned}$$

In the above, one goes from the first to the second line by realizing that for a fixed unit vector $\boldsymbol{\omega}$ in $\mathbb{R}^2$, the equation

$$\mathbf{x}\cdot\boldsymbol{\omega} = \sigma$$

is the equation of the line in $\mathbb{R}^2$ with normal vector $\boldsymbol{\omega}$ and oriented distance σ from the origin. So the notation $ds_{\boldsymbol{\omega},\sigma}$ denotes the arc length along the line determined by the unit vector $\boldsymbol{\omega}$ and the number σ.

Now apply the inversion formula. Then

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \widehat{f}(\boldsymbol{\xi})e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\,d\boldsymbol{\xi} \\ &= \frac{1}{2\pi}\int_0^{+\infty} r\,dr \int_{\mathbb{S}^1} \widehat{f}(r\boldsymbol{\omega})e^{ir\mathbf{x}\cdot\boldsymbol{\omega}}\,ds_{\mathbb{S}^1}(\boldsymbol{\omega}) \\ &= \frac{1}{(2\pi)^2}\int_0^{+\infty} r\,dr \int_{\mathbb{S}^1} e^{ir\mathbf{x}\cdot\boldsymbol{\omega}}\,ds_{\mathbb{S}^1}(\boldsymbol{\omega}) \int_{-\infty}^{+\infty} e^{-ir\sigma}\,d\sigma \int_{\mathbf{x}\cdot\boldsymbol{\omega}=\sigma} f(\mathbf{x})e^{-ir\mathbf{x}\cdot\boldsymbol{\omega}}\,ds_{\boldsymbol{\omega},\sigma}(\mathbf{x}). \end{aligned}$$

So $f(\mathbf{x})$ is represented by the values of its integrals over lines in the plane, which was our claim.
qed

Remark 1. If one wishes to apply the above argument to 3–space, $\mathbb{R}^3$, one would obtain $f(\mathbf{x})$ as determined by integrals over 2–planes in $\mathbb{R}^3$. But the usual x–ray equipment can only calculate integrals over lines. So one needs a more sophisticated argument.