III. Fourier Series

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§1. Fourier series and their coefficients

We are given the L–periodic, piecewise continuous complex-valued function $\phi(x)$, for x in $(-\infty, +\infty)$, and we wish to represent $\phi(x)$ by a series of the form

$$\phi(x) = \sum_{n=-\infty}^{+\infty} a_n e^{2\pi i n x/L}. \tag{1}$$

Of course, for each integer n, the function

$$x \mapsto e^{2\pi i n x/L}$$

is L–periodic. So any finite linear combination of such functions is also L–periodic. Our interest, therfore, is in determining to what extent the above collection of functions is the basic building block — by "infinite linear combinations" — of all L–periodic functions.

To find a *candidate for a_n,* we formally calculate:

$$\int_0^L \phi(x) e^{-2\pi i n x/L}\, dx \quad = \quad \int_0^L \left\{ \sum_{k=-\infty}^{+\infty} a_k e^{2\pi i k x/L} \right\} e^{-2\pi i n x/L}\, dx$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{+\infty} a_k \int_0^L e^{-2\pi i(n-k)x/L}\,dx \\
&= a_n L;
\end{aligned}
$$

so the natural choice for a_n is

$$
a_n = \frac{1}{L}\int_0^L \phi(x) e^{-2\pi inx/L} dx. \tag{2}
$$

The basic question is: to what extent is this choice of a_n, for any given ϕ, justified? Well, if $\phi(x)$ is actually given by the *finite* sum

$$
\phi(x) = \sum_{n=-N}^{N} \alpha_n e^{2\pi inx/L},
$$

then for a_n given by (1) we certainly have

$$
a_k = \alpha_k
$$

for $k = -N, \ldots, N$, and all other a_k vanish. Therefore a more precise version of the question is:

Given arbitrary ϕ, to what extent is it true that the function

$$
s_N(x;\phi) = \sum_{n=-N}^{N} a_n e^{2\pi inx/L}, \qquad a_n = \frac{1}{L}\int_0^L \phi(x) e^{-2\pi inx/L} dx
$$

is close to the function ϕ, for large N? The answer is: It depends on what one means by "close." Here are three possibilities:

(a) **pointwise convergence:** For a given x in $(-\infty, +\infty)$, we have

$$
\lim_{N\to+\infty} s_N(x;\phi) = \phi(x).
$$

(b) **mean-square convergence:**

$$
\lim_{N\to+\infty} \int_0^L \{\phi(x) - s_N(x;\phi)\}^2\, dx = 0.
$$

(c) **uniform convergence:**

$$
\lim_{N\to+\infty} \sup_x |\phi(x) - s_N(x;\phi)| = 0.
$$

NOTE: For students who had advanced calculus, they already know the definition of the *supremum*. In our context of periodic finunctions, if a function is continuous, then the supremum is the same as the maximum value of the function. If the function is allowed to have jump discontinuities then the supremum refers to the maximum value of right and left hand limits of the function.

It is easy to show that

Theorem 1. *Uniform convergence implies the other two.*

Proof. Certainly uniform convergence implies pointwise convergence. That uniform convergence implies mean-square convergence follows from the estimate

$$\int_0^L \{\phi(x) - s_N(x;\phi)\}^2 dx \le L\{\sup_x |\phi(x) - s_N(x;\phi)|\}^2. \qquad \text{qed}$$

Example 1. Let $f_n(x)$ be a function on the real line for which

(i) $f_n(x) = 0$ for all $x \le 0$ and for all $x \ge 1/n$;

(ii) $f_n(1/2n) = 1$;

(iii) $f_n(x)$, for $0 \le x \le 1/n$ consists of the line segments from $(0,0)$ to $(1/2n, 1)$, and from $(1/2n, 1)$ to $(1,0)$.

(Reader! Draw yourself a picture of the function.) Then

$$\lim_{n\uparrow+\infty} f_n(x) = 0 \text{ for every single } x; \qquad \text{but} \qquad \sup_x |f_n(x) - 0| = 1$$

— so f_n converges, but not uniformly. Also note that f_n mean-square converges to 0.

We, oddly enough, start by studying a different kind of convergence — the uniform convergence of Caesaro sums of ϕ to ϕ itself. By a **Caesaro sum** we mean

$$\delta_N(x;\phi) := \frac{1}{N}\sum_{k=0}^{N-1} s_k(x;\phi)$$

— the *average* of the first N partial sums of the Fourier series. The naive idea is that irregularities in the original sequence of partial sums, that might prevent its convergence, become muted when passing to the sequence of averages. It is a standard fact that

Theorem 2. *Ordinary convergence of $s_N(x;\phi)$ implies convergence of $\delta_N(x;\phi)$.*

The key to study of both $s_N(x;\phi)$ and $\delta_N(x;\phi)$ are the **Dirichlet** and **Fejer** kernels of Example II.1. There, in Example II.1 the Dirichlet and Fejer kernels are defined to be 2π–periodic, so we shall redefine them *here* to be L–periodic, namely,

$$\mathbf{D}_N(\xi) := \frac{1}{L}\sum_{n=-N}^{N} e^{2\pi inx/L} = \frac{1}{L}\frac{\sin 2\pi(N+1/2)\xi/L}{\sin \pi\xi/L}, \tag{3}$$

$$\mathbf{F}_N(\xi) := \frac{1}{N}\sum_{k=0}^{N-1} \mathbf{D}_N(\xi) = \frac{1}{NL}\left\{\frac{\sin \pi N\xi/L}{\sin \pi\xi/L}\right\}^2. \tag{4}$$

Theorem 3. *The partial and Caesaro sums of the Fourier series of ϕ are given by*

$$s_N(x;\phi) = \int_{-L/2}^{L/2} \mathbf{D}_N(\xi - x)\phi(\xi)\,d\xi \tag{5}$$

$$\delta_N(x;\phi) = \int_{-L/2}^{L/2} \mathbf{F}_N(\xi - x)\phi(\xi)\,d\xi. \tag{6}$$

Proof. Well,

$$\int_{-L/2}^{L/2} \mathbf{D}_N(\xi - x)\phi(\xi)\,d\xi = \int_{-L/2}^{L/2} \frac{1}{N}\sum_{n=-N}^{N} e^{2\pi in(\xi-x)/L}\phi(\xi)\,d\xi = \cdots$$

$$\cdots = \sum_{n=-N}^{N} e^{-2\pi inx/L}\frac{1}{N}\int_{-L/2}^{L/2} e^{2\pi in\xi/L}\phi(\xi)\,d\xi = \sum_{n=-N}^{N} a_{-n}e^{-2\pi inx/L} = s_N(x;\phi),$$

which implies (5). Then (6) also follows immediately. qed

In Example II.1 we showed that

$$\lim_{N\uparrow+\infty} \mathbf{F}_N(\xi - x) = \delta_x(\xi).$$

Therefore, *if ϕ is L–periodic, and continuous at all x, then $\delta_N(x;\phi) \to \phi(x)$ as $N \uparrow +\infty$.* Actually more is true. The argument of Theorem II.1 also shows:

Theorem 4. *If ϕ is L–periodic and continuous at all x, then $\delta_N(\ ;\phi) \to \phi$ uniformly.*

Remark 1. The fact that the Dirichlet kernel is *not* an approximation to the identity, in the sense of Theorem II.1, is what makes the question of convergence of partial sums more subtle than convergence of Caesaro sums. One sees, rather strikingly, the improvement in control achieved by passing from the partial sums to the Caesaro sums.

Remark 2. It is helpful to think of the formulae (1) and (2) as defining a transform and its inverse. Namely, one direction is to consider a mapping from L–periodic functions on the line to functions on the integers $\phi(x) \mapsto \widehat{\phi}(n)$, given by

$$\widehat{\phi}(n) = \frac{1}{L}\int_0^L \phi(x)e^{-2\pi inx/L}dx.$$

So $\widehat{\phi}(n)$ is the n–th Fourier coefficient. The opposite direction, the "inverse transform" is to associate to the function $a(n)$ on the integers the L–periodic function

$$\phi(x) = \sum_{n=-\infty}^{+\infty} a(n)e^{2\pi inx/L}.$$

Our study then is to determine to what extent the "inverse transform" is a genuine inverse.

Exercises

Exercise 1. Show that

$$\int_{-\pi}^{\pi} \cos\, nx \cos\, kx\, dx = \int_{-\pi}^{\pi} \sin\, nx \sin\, kx\, dx = 0$$

for all integers $n \neq k$; that

$$0 = \int_{-\pi}^{\pi} \cos\, nx \sin\, kx\, dx$$

for all integers n, k; and that

$$\pi = \int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \sin^2 nx\, dx$$

for all integers n.

Exercise 2. Give the formal calculation to show that if the complex-valued 2π–periodic function $f(x)$ has the Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos\, nx + b_n \sin\, nx\},$$

then

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

for all integers $n = 0, 1, 2, \dots$ and $k = 1, 2, \dots$.

Exercise 3. Given the complex-valued 2π–periodic function $f(x)$ with Fourier series expansions

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx} = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\},$$

then write the coefficients c_n in terms of a_n, b_k, and vice versa.

Exercise 4. Show that $a_n = 0$ for all n when the function $f(x)$ is an odd function, that is, $f(x) = -f(-x)$ for all x. Similarly, show that $b_k = 0$ for all k when the function $f(x)$ is an even function, that is, $f(x) = f(-x)$ for all x.

Exercise 5. Expand the function $f(x) = Ax^2 + Bx + C$, where $-\pi < x < \pi$, for the constants A, B, C in a Fourier series. (Note: The function is only given for the interval $(-\pi, \pi)$. The idea is to then extend the function to the full real line by 2π–periodicity.)

Exercise 6. Expand the following functions in Fourier series:

(a) $f(x) = e^{ax}$ $(-\pi < x < \pi)$, where $a \neq 0$ is a constant;

(b) $f(x) = \cos ax$ $(-\pi < x < \pi)$, where a is not an integer;

(c) $f(x) = \sin ax$ $(-\pi < x < \pi)$, where a is not an integer.

Exercise 7.

(a) Let y be a real number, ϕ a function on $(-\infty, +\infty)$. Let $\tau_y \phi$ denote *the translate of ϕ by y,* given by

$$(\tau_y \phi)(x) = \phi(x - y).$$

Show that if ϕ is L–periodic piecewise continuous then

$$\widehat{\tau_y \phi}(n) = e^{-2\pi i n y / L} \widehat{\phi}(n).$$

(b) Also show that

$$\widehat{\phi}(n - m) = \{e^{imx} \phi\}\widehat{\ }(n).$$

(c) Given the L–periodic piecewise continuous functions ϕ, ψ, consider their convolution

$$(\phi * \psi)(x) = \frac{1}{L} \int_0^L \phi(x - y) \psi(y) \, dy,$$

and show that

$$\widehat{\phi}(n)\widehat{\psi}(n) = \widehat{\phi * \psi}(n).$$

The analogy to the Laplace transform should be clear.

§2. Mean-square convergence

Here we are still working with our collection of L–periodic functions on the real line. This collection of functions forms a vector space over the complex numbers.

Definition. For any two such functions ϕ, ψ, we define (an analog of the "dot product," commonly called, in our context) the *inner product of ϕ and ψ*, by

$$(\phi, \psi) := \frac{1}{L}\int_0^L \phi(x)\overline{\psi}(x)\,dx.$$

Then for constants α_1, α_2 and functions $\phi, \phi_1, \phi_2, \psi$ we have

$$\begin{aligned} (\alpha_1\phi_1 + \alpha_2\phi_2, \psi) &= \alpha_1(\phi_1, \psi) + (\alpha_2\phi_2, \psi), \\ (\phi, \psi) &= \overline{(\psi, \phi)}, \\ (\phi, \phi) &\geq 0. \end{aligned}$$

We have $(\phi, \phi) = 0$ if and only if $\phi = 0$ for all but, at most a finite number of x in any bounded interval.

Definition. For any ϕ we define the *norm of ϕ* by

$$|\phi| = \sqrt{(\phi, \phi)}.$$

Then for a constant α and a function ϕ we have

$$\|\alpha\phi\| = |\alpha|\|\phi\|.$$

One has the **Cauchy–Schwarz inequality**:

$$|(\phi, \psi)| \leq \|\phi\|\|\psi\|,$$

with equality if and only if there exist constants α and β such that

$$\alpha\phi + \beta\psi = 0$$

for all but, at most, a finite number of x in any bounded interval. One then easily derives the **triangle inequality**:

$$\|\phi + \psi\| \le \|\phi\| + \|\psi\|.$$

Definition. Given ϕ, ψ, we may define their *mean-square distance* to be equal to $\|\phi - \psi\|$. Note that

$$\|\phi - \psi\| = \|\psi - \phi\|; \qquad \|\phi - \psi\| \ge 0,$$

with equality if and only if $\phi = \psi$ at all but, at most, a finite of points in any bounded interval; and

$$\|\phi - \psi\| \le \|\phi - \sigma\| + \|\sigma - \psi\|$$

for functions ϕ, ψ, σ.

Definition. A collection of functions $\{\phi_1, \phi_2, \ldots\}$ (whether the collection is finite or infinite) is said to be *orthogonal* if

$$(\phi_j, \phi_k) = 0 \qquad \text{for all } j \ne k.$$

The collection is said to be *orthonormal* if, in addition to being orthogonal, we also have

$$\|\phi_k\| = 1 \qquad \text{for all } k.$$

Theorem 5. *If the collection $\{\phi_1, \ldots, \phi_n\}$ is orthonormal, and*

$$\phi = \sum_{k=1}^{n} c_k\phi_k, \qquad \psi = \sum_{k=1}^{n} d_k\phi_k,$$

then

$$(\phi, \psi) = \sum_{k=1}^{n} c_k\overline{d}_k, \qquad \|\phi\| = \sum_{k=1}^{n} |c_k|^2.$$

In particular,

(7) $$c_k = c_k(\phi) = (\phi, \phi_k).$$

Proof. Straightforward calculation. qed

Remark 1. Therefore, if $\{\phi_1, \dots, \phi_n\}$ are orthonormal, then they are linearly independent.

We now consider the **least-squares approximation**. Given the finite orthonormal collection $\{\phi_1, \dots, \phi_n\}$. Suppose we are given ϕ and we seek the best choice of numbers $\alpha_1, \dots, \alpha_n$ to make

$$\|\phi - \sum_{k=1}^{n} \alpha_k \phi_k\|$$

as small as possible. Well, for $c_k(\phi)$ given in (7) we have

$$\begin{aligned}
\|\phi - \sum_{k=1}^{n} \alpha_k \phi_k\|^2 &= (\phi - \sum_{k=1}^{n} \alpha_k \phi_k, \phi - \sum_{\ell=1}^{n} \alpha_\ell \phi_\ell) \\
&= \|\phi\|^2 + \sum_{k=1}^{n} \{|\alpha_k|^2 - 2\mathrm{Re}\, \alpha_k \overline{c_k}(\phi)\} \\
&= \|\phi\|^2 + \sum_{k=1}^{n} \{|\alpha_k|^2 - 2\mathrm{Re}\, \alpha_k \overline{c_k}(\phi) + |c_k(\phi)|^2 - |c_k(\phi)|^2\} \\
&= \|\phi\|^2 - \sum_{k=1}^{n} |c_k(\phi)|^2 + \sum_{k=1}^{n} |\alpha_k - c_k(\phi)|^2.
\end{aligned}$$

In summary,

$$\|\phi - \sum_{k=1}^{n} \alpha_k \phi_k\|^2 = \|\phi\|^2 - \sum_{k=1}^{n} |c_k(\phi)|^2 + \sum_{k=1}^{n} |\alpha_k - c_k(\phi)|^2. \tag{8}$$

The $c_k(\phi)$'s are fixed, and the α_k's vary freely. The best way to make the left hand side of (8) as small as possible is to pick

$$\alpha_k = c_k(\phi) \qquad \text{for all } j. \tag{9}$$

Theorem 6. *We therefore have*

$$\|\phi - \sum_{k=1}^{n} \alpha_k \phi_k\| \geq \|\phi - \sum_{k=1}^{n} c_k(\phi) \phi_k\| \tag{10}$$

for all possible $\{\alpha_1, \dots, \alpha_n\}$*, with equality if and only if* (9) *is valid.*

An elementary, but very important, consequence of the above calculation is the following.

Theorem 7. *If* α_k *is chosen as in* (9)*, then we have* **Bessel's inequality***:*

$$\sum_{k} |c_k(\phi)|^2 \leq \|\phi\|^2 < +\infty. \tag{11}$$

Proof. From (8)

$$\|\phi\|^2 - \sum_{k=1}^{n} |c_k(\phi)|^2 = \|\phi - \sum_{k=1}^{n} c_k(\phi)\phi_k\|^2 \geq 0. \tag{12}$$

So

$$\|\phi\|^2 \geq \sum_{k=1}^{n} |c_k(\phi)|^2, \qquad \text{for all } \ n.$$

Therefore, if we are given an infinite orthonormal sequence $\{\phi_k\}$, then (11) follows. qed

One naturally asks: Is there equality in Bessel's inequality?

Example 1. Consider L–periodic functions on the lines, and the orthonormal sequence

$$\phi_n(x) = e^{2\pi inx/L} \text{ for } n = 0, +1, -1, +2, -2, \ldots$$

So

$$c_n(\phi) = \frac{1}{L}\int_0^L \phi(x)e^{-2\pi inx/L}\,dx = \widehat{\phi}(n). \tag{13}$$

The basic result is

Theorem 8. Parseval's theorem. *We always have mean-square convergence of* $s_N(\ ;\phi)$ *to* ϕ*, for piecewise continuous* ϕ*, namely,*

$$0 = \lim_{N\to+\infty} \|\phi - s_N(\ ;\phi)\|^2 = \lim_{N\to+\infty} \|\phi\|^2 - \sum_{n=-N}^{N} |c_n(\phi)|^2. \tag{14}$$

In particular,

$$\|\phi\|^2 = \sum_{n=-\infty}^{+\infty} |c_N(\phi)|^2. \tag{15}$$

Proof. We want to show that $\|\phi - s_n(\ ;\phi)\|^2$ becomes small when n becomes large. We accomplish this in two stages:

Step 1. Given any N, we have for all $n \geq N$,

$$\|\phi - s_n(\ ;\phi)\|^2 \quad = \quad \|\phi\|^2 - \sum_{k=-n}^{n} |c_k(\phi)|^2$$

$$
\begin{aligned}
&\le \|\phi\|^2 - \sum_{k=-N}^{N} |c_k(\phi)|^2 \\
&= \|\phi - \sum_{k=-N}^{N} c_k(\phi)\phi_k\|^2 \\
&\le \|\phi - \sum_{k=-N}^{N} \alpha_k\phi_k\|^2
\end{aligned}
$$

for all choices of $\alpha_{-N}\ldots\alpha_N$. So as soon as we find a choice of N; $\alpha_{-N}\ldots\alpha_N$ for which $\|\phi - \sum_{k=-N}^{N} \alpha_k\phi_k\|$ is small, we are automatically guaranteed that $\|\phi - s_n(\ ;\phi)\|$ is at least as small for all $n \geq N$.

Step 2. How to find such a choice of N; $\alpha_{-N}\ldots\alpha_N$. If ϕ is continuous for all x, then we use the Caesaro sums $\delta_N(x;\phi)$ to approximate $\phi(x)$. The Caesaro sum is a certainly linear combination of $\phi_{-N},\ldots,\phi_N$; and $\delta_N(\ ;\phi) \to \phi$ uniformly. But uniform convergence implies mean-square convergence, which is what we want.

If ϕ is only piecewise continuous, then one easily approximates (in the mean-square distance) ϕ by everywhere continuous ψ. Then ψ is approximated by its sequence of Caesaro sums. qed

Exercises

Exercise 8. Suppose that $\{\phi_n\}$ is any *complete orthonormal system,* that is, for any ϕ, we have the Parseval theorem:

$$0 = \lim_{N\to\infty} \|\phi - s_N(\ ;\phi)\|.$$

Show that for any given ϕ, ψ we have

$$(\phi,\psi) = \sum_n c_n(\phi)\overline{c_n(\psi)}.$$

HINT: Give a calculation for the inner product

$$(\phi - s_n(\ ;\phi), \psi).$$

Show that, on the other hand, that the Cauchy–Schwartz inequality, and Parseval's theorem, imply

$$\lim_{N\to+\infty} (\phi - s_N(\ ;\phi), \psi) = 0.$$

Apply this last result to your calculation.

Exercise 9. Give the formal Fourier series of δ_y, the Dirac delta function concentrated at y. Show that the Fourier series can be only a formal expression, i.e., show

$$\sum_n |c_n(\delta_y)|^2 = +\infty.$$

For $h > 0$, construct an approximation to the identity, calculate its Fourier coefficients, and see what happens when h goes to 0.

Exercise 10. Weyl's theorem.

(a) Let $f(x)$ be continuous 2π–periodic. Show:

$$\lim_{t\uparrow+\infty} \frac{1}{t}\int_0^t f(x_0+s)\,ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

for all x_0.

(b) Now we replace t by a discrete variable. Namely, let $\alpha \in (0,1)$, and consider

$$\mathcal{F}_n(x_0) := \frac{1}{n}\sum_{k=0}^{n-1} f(x_0 + 2\pi k\alpha).$$

(This is the analogue of the left hand side above.)

(i) Show that if α is rational, then one can construct a counterexample to:

$$\lim_{n\uparrow+\infty} \mathcal{F}_n(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{16}$$

(ii) Give the formal calculation to prove (16) when α is irrational.

(iii) Prove (16) when α is irrational. HINT: First, just give a formal calculation to get the idea of the result. Then, to give a more precise version, let δ_N denote the n–th Caesaro approximation of f, and show that it suffices to prove the result for δ_N. Then, prove the claim for δ_N.

§3. Uniform convergence of Fourier series

Recall, given a series

$$\sum_n a_n,$$

we say that the series is *absolutely convergent* if

$$\sum_n |a_n| < +\infty.$$

If a series converges absolutely, then it converges in the usual sense.

If given an interval I, and the series of functions

$$\sum_n f_n(x), \tag{17}$$

each defined on I, then we say that the series (17) *converges uniformly* if

$$\lim_{N\to+\infty}\{\sup_I |F(x) - \sum_{|n|\leq N} f_n(x)|\} = 0.$$

The main results

Theorem 9. *If the Fourier series of an L–periodic piecewise continuous function ϕ converges uniformly, then its limit is equal to ϕ itself for all but, at most, a finite number of points in any bounded interval.*

We now consider situations in which one concludes that the Fourier series converges uniformly. Bessel's inequality says, that for ϕ an L–periodic piecewise continuous function, that $c_k(\phi)$ converges to 0 quickly enough to guarantee that

$$\sum_k |c_k(\phi)|^2 < +\infty.$$

Is there more information? We shall discover the striking fact that the speed with which $c_k(\phi) \to 0$ is related to the uniform convergence of $s_n(\ ;\phi)$ to ϕ, and the smoothness of ϕ.

Definition. We let C^k, for $k \geq 1$, denote the set of functions which have k continuous derivatives. When $k = 0$, then C^0 denotes the collection of continuous functions.

Theorem 10. *If ϕ is L–periodic continuous, with ϕ' piecewise continuous, then the Fourier series of ϕ converges absolutely uniformly to ϕ. Also, the Fourier coefficients of ϕ' are given by*

$$c_n(\phi') = 2\pi i n c_n(\phi)/L. \tag{18}$$

More generally, if ϕ is L–periodic continuous C^k, $k \geq 0$, with piecewise continuous $\phi^{(k+1)}$, then

$$c_n(\phi^{(\ell)}) = (2\pi i n/L)^\ell c_n(\phi)$$

for all $\ell = 1, 2 \dots k+1$, *and*

$$n^{k+1} c_n(\phi) \to 0 \tag{19}$$

as $n \uparrow +\infty$. *Furthermore, for every* $j = 1 \dots k$, *the Fourier series of* $\phi^{(j)}$ *converges absolutely uniformly to* $\phi^{(j)}$.

So given information on the smoothness of the function ϕ one deduces information on the speed with which the Fourier coefficients converge to 0.

What can be said conversely, that is, in the other direction? That is, if one has information on the speed with which the Fourier coefficients of ϕ converge to 0, what can one say about the smoothness of ϕ?

Theorem 11. *Given the Fourier series*

$$\sum_{n=-\infty}^{+\infty} a_n e^{2\pi i n x/L}, \tag{20}$$

which satisfies

$$|a_n| \leq \frac{\text{const.}}{|n|^{k+1+\epsilon}}, \tag{21}$$

for a given positive integer k, *and for all integers* n, *then it converges absolutely uniformly to a* C^k *function* ϕ. *The* k *derivatives of* ϕ *may be obtained through term-by-term differentiation of* (20), *and in each case, the new Fourier series converges absolutely uniformly to the corresponding derivative.*

This concludes the discussion of the results. Before proceeding with the proofs,it is best to give a number of exercises emphasizing and illustrating the results themselves.

Exercises

Exercise 11. Separation of variables for the heat equation. Let x range over the real numbers, and t over the positive real numbers. Given an L–periodic function $\phi(x)$, we wish to solve the *initial value problem for the heat equation on the circle*, namely we wish to solve, for $u = u(x,t)$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad u(x+L,t) = u(x,t) \ \text{ for all } \ x, t, \qquad \lim_{t \downarrow 0} u(x,t) = \phi(x) \ \text{ for all } \ x.$$

The method consists of first finding a selected collection of solutions to the heat equation satisfying the boundary conditions, and then working in the initial data.

(a) To solve the heat equation first consider solutions $u(x,t)$ of the form

$$u(x,t) = T(t)X(x).$$

Then substitute into the heat equation $u_{xx} = u_t$ to obtain

$$T(t)X''(x) = T'(t)X(x),$$

where each prime refers to differentiation with respect to the indicated variable, whixh implies

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

But each side depends on different variables; therefore the two sides must be constant, that is, there exists a constant λ such that

$$X''(x) + \lambda X(x) = 0, \qquad T'(t) + \lambda T(t) = 0.$$

The second equation implies that λ is real. The solutions to the above equations, for $X(x)$, are linear combinations of solutions of the form

$$X(x)(= \begin{cases} Ae^{-\sqrt{\lambda}x} + Be^{\sqrt{\lambda}x} & \lambda > 0 \\ A + Bx & \lambda = 0 \\ A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x & \lambda < 0. \end{cases}$$

To determine which of these yields a possible solution, consider the PERIODIC CONDITION

$$X(x+L) = X(x) \quad \text{for all} \quad x.$$

Show that the only possibility is that $X(x)$ have two zeroes is when λ is nonnegative, in which case we have

$$\lambda = -4\pi^2 n^2/L^2, \qquad X(x) = A\cos(2\pi n/L)x + B\sin(2\pi n/L)x,$$

or, still better

$$\lambda = -4\pi^2 n^2/L^2, \qquad X(x) = \alpha_n e^{(2\pi i n/L)x} + \alpha_{-n}e^{-(2\pi i n/L)x},$$

for $n = 0, 1, \ldots$. (Of course, when $n = 0$ one only has the constant term.) This implies that

$$T(t) = e^{-(4\pi^2 n^2/L^2)t};$$

So the most ambitious solution obtained by this approach is an infinite linear combination, that is, a series written as

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{-(4\pi^2 n^2/L^2)t} e^{2\pi i n x/L}.$$

The next question is: what values to pick for the coefficients α_n? Use the INITIAL CONDITIONS. For $t = 0$ we have

$$\phi(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inx/L}.$$

So the proper choice of coefficients a_n will be the Fourier coefficients of ϕ: $a_n = c_n(\phi)$.

(b) Verify that the series

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(\phi) e^{-(4\pi^2 n^2/L^2)t} e^{2\pi inx/L}$$

converges uniformly in x, and in $t \in [\alpha, \beta] \subset (0, \infty)$. Show that one may differentiate the series term-by-term to verify that $u(x,t)$ satisfies the heat equation.

Verifying that $\lim_{t\downarrow 0} u(x,t) = \phi(x)$ for all x is not so obvious. Assume, nonetheless, that it is true.

(c) Show that $u(x,t)$ given above can be written as

$$u(x,t) = \int_0^L p(x,y,t)\phi(y)\,dy,$$

where

$$p(x,y,t) = \sum_{n=-\infty}^{\infty} e^{-(4\pi^2 n^2/L^2)t} e^{2\pi inx/L} e^{-2\pi iny/L}.$$

So we may write

$$\lim_{t\downarrow 0} p(x,y,t) = \delta_x(y).$$

Exercise 12. Poisson's integral formula. Consider the unit disk **D** in $\mathbb{C}$, with center located at the origin. Given a 2π periodic function ϕ, we think of ϕ as defined on the unit circle $\mathbb{S}$, the boundary of **D**. We want to solve the **Dirichlet problem for harmonic functions** on **D**, namely, we seek a **harmonic** function $u = u(z)$, $z = x + iy$, that is a solution to **Laplace's equation**

$$\Delta u := u_{xx} + u_{yy} = 0, \tag{22}$$

satisfying the boundary condition

$$\lim_{z\to w} u(z) = \phi(w), \tag{23}$$

for all w in $\mathbb{S}$.

(a) Write z in polar coordinates

$$z = re^{i\theta},$$

and derive the formula

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

for the Laplacian in polar coordinates.

(b) Next carry out a separation of variables argument in this context, that is, look for solutions of (22) of the form

$$u(r,\theta) = R(r)\Theta(\theta), \tag{24}$$

and show that such a solution of (22) must satisfy

$$R'' + (1/r)R' - (\lambda/r^2)R = 0, \qquad \Theta'' + \lambda\Theta = 0,$$

for some constant λ.

(c) Show that the full collection of solutions of (22) of the form (24) are given by

$$u(r,\theta) = r^{|n|}e^{in\theta},$$

where n ranges over all the integers.

(d) Given ϕ, propose a solution for (22):(23), and show it may be written as

$$u(r,\theta) = (1/2\pi)\int_{-\pi}^{\pi} \frac{\phi(e^{i\alpha})(1-r^2)}{1-2r\cos(\alpha-\theta)+r^2}\,d\alpha.$$

HINT: Naturally, the proposed solution is a series

$$u(r,\theta) = \sum_{n=-\infty}^{+\infty} a_n r^{|n|}e^{in\theta},$$

and the issue is how to pick the coefficients a_n. Of course, set $r = 1$; then a_n is the n–th Fourier coefficient of ϕ. Substitute in the formula for a_n, use the geometric series, and do a little of this and a little of that.

(e) Prove that if ϕ is continuous at $w = e^{i\theta}$, then $u(r,\theta) \to \phi(e^{i\theta})$ as $r \uparrow 1$. HINT: It suffices to show that the family

$$\mathcal{K}_r(\alpha) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\alpha+r^2}$$

is an approximation to the identity on the circle, as $r \uparrow 1$.

Exercise 13. Bessel functions.

(a) For $\nu \geq 0$ and complex z, set

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)},$$

and show that $J_\nu(z)$ satisfies the differential equation

$$x^2{J_\nu}'' + x{J_\nu}' + (x^2-\nu^2)J_\nu = 0.$$

(b) Consider r the variable, z a parameter, expand

$$e^{(r-1/r)z/2} = e^{zr/2}e^{-z/2r} = \sum_{n=-\infty}^{\infty} a_n(z)r^n,$$

in a Laurent series (or a Z–transform) in r, and calculate the coefficients

$$n \geq 0 \qquad \Rightarrow \qquad a_n(z) = J_n(z),$$

$$n < 0 \qquad \Rightarrow \qquad a_n(z) = (-1)^n J_{|n|}(z).$$

(c) Set $r = e^{i\theta}$ and derive the integral representation

$$a_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(z \sin\, \theta - n\theta)}\, d\theta.$$

Exercise 14. Diffusion on integers. Instead of functions on the real line, consider functions on the integers. So $f = f(n)$. The Laplacian here is given by

$$(\Delta f)(n) = \frac{f(n+1) + f(n-1)}{2} - f(n).$$

For $u = u(n,t)$, where n ranges over the integers, and t ranges over positive time, consider the heat equation

$$\Delta u = \frac{\partial u}{\partial t}, \qquad t > 0,$$

with initial-values given by

$$\lim_{t \downarrow 0} u(n,t) = \phi(n),$$

where $\phi(n)$ is a given function on the integers.

(a) Think of $u(n,t)$ as Fourier coefficients depending on time, that is, consider the function on the circle, having time t as parameter,

$$U(\theta, t) = \sum_{n=-\infty}^{\infty} u(n,t) e^{in\theta},$$

and derive the equation

$$\frac{\partial U}{\partial t}(\theta, t) = \{\cos\,\theta\ - 1\} U(\theta, t).$$

(b) Show that

$$U(\theta, t) = e^{-(1-\cos\,\theta)t} \sum_{n=-\infty}^{\infty} \phi(n) e^{in\theta}.$$

(c) Now show that one obtains

$$u(n,t) = i^{-n} e^{-t} \sum_{k=-\infty}^{\infty} \phi(k) J_{n-k}(it)$$

for the solution to the initial value problem.

Exercise 15. Wirtinger's inequality. Prove that if f is continuous L–periodic, f' is piecewise continuous, and

$$\int_0^L f(\xi)\,d\xi = 0,$$

then

$$\int_0^L |f'|^2 \geq \frac{4\pi^2}{L^2}\int_0^L |f|^2,$$

with equality if and only if

$$f(x) = a_{-1}e^{-2\pi ix/L} + a_1 e^{2\pi ix/L}.$$

Exercise 16. The isoperimetric inequality. Prove that if C is a continuous, piecewise differentiable, simple closed curve, of length L, enclosing a domain D, of area A, then

$$L^2 \geq 4\pi A,$$

with equality if and only if C is a circle. SKETCH: Start with Green's Theorem to show

$$\begin{aligned} 2A &= \int_C x\,dy - y\,dx \\ &= \int_C \mathbf{r}\cdot\nu\,ds, \end{aligned}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, ν is the outward unit normal vector field along C, and ds is arc length along C. Therefore

$$\begin{aligned} 2A &= \int_C \mathbf{r}\cdot\nu\,ds \\ &\leq \int_C |\mathbf{r}||\nu|\,ds \\ &= \int_C |\mathbf{r}|\,ds \\ &\leq \left\{\int_C |\mathbf{r}|^2\,ds\right\}^{1/2}\left\{\int_C 1^2\,ds\right\}^{1/2} \\ &\leq \left\{\frac{L^2}{4\pi^2}\int_C |\mathbf{r}'|^2\,ds\right\}^{1/2} L^{1/2} \\ &= \frac{L^{3/2}}{2\pi}\left\{\int_C |\mathbf{r}'|^2\,ds\right\}^{1/2}. \end{aligned}$$

Assume C is parametrized with respect to arclength. Then

$$2A \leq \frac{L^{3/2}}{2\pi}\left\{\int_C ds\right\}^{1/2} = \frac{L^2}{2\pi},$$

which is the desired inequality. Validate the above argument. Note that the last inequality in the above chain is *ostensibly* Wirtingers's inequality. But we require (in order to apply Wirtinger's inequality)

$$\int_C \vec{r}\, ds = \vec{0}.$$

Is this a real difficulty? Of course not — why not?

Prove the case of equality.

The proofs

First recall two facts about uniform convergence.

If each f_n is continuous on the interval I, and the series (17) converges uniformly to F on I, then it is a fact that F is also continuous on I.

If, in the above paragraph, the interval I is closed and bounded, then one also has

$$\sum_n \int_I f_n(x)\, dx = \int_I F(x)\, dx.$$

Theorem 9. *If the Fourier series of an L–periodic piecewise continuous function ϕ converges uniformly, then its limit is equal to ϕ itself for all but, at most, a finite number of points in any bounded interval.*

Proof. Assume

$$\lim_{N\to+\infty} s_N(x;\phi) = \psi(x)$$

for some function $\psi(x)$, uniformly; we want to show $\phi = \psi$ for all but, at most, a finite number of points in any bounded interval. Well, ψ is L–periodic and continuous. Also,

$$\begin{aligned} c_n(\psi) &= \frac{1}{L}\int_0^L \psi(x)e^{-2\pi inx/L}\, dx \\ &= \frac{1}{L}\int_0^L \left\{ \sum_{k=-\infty}^{+\infty} c_k(\phi)e^{2\pi ikx/L} \right\} e^{-2\pi inx/L}\, dx \\ &= \sum_{k=-\infty}^{+\infty} \frac{c_k(\phi)}{L} \int_0^L e^{-2\pi i(n-k)x/L}\, dx \\ &= c_n(\phi), \end{aligned}$$

that is, $c_n(\psi) = c_n(\phi)$; so

$$s_N(\ ;\psi) = s_N(\ ;\phi) \tag{25}$$

for all N. The triangle inequality, the Parseval theorem, and (25), imply

$$\|\phi - \psi\| = \|\phi - s_N(\ ;\phi) + s_N(\ ;\psi) - \psi\| \le \|\phi - s_N(\ ;\phi)\| + \|s_N(\ ;\psi) - \psi\| \to 0$$

as $N \to +\infty$. So $\|\phi - \psi\| = 0$, which implies the theorem. qed

We require two more preliminaries for the proof of Theorem 10.

The **Weierstrass** M**–test** states that if there is a sequence M_n of nonnegative real numbers satisfying

$$|f_n(x)| \le M_n \quad \text{for all} x \in I, \qquad \text{and} \quad \sum_n M_n < +\infty,$$

then the series (17) converges absolutely uniformly to some function F in I.

Our next preliminary is the following elementary remark. For $a, b > 0$ one has

$$0 \le (a-b)^2 = a^2 + b^2 - 2ab,$$

so

$$ab \le \{a^2 + b^2\}/2.$$

Therefore, given a series

$$\sum_n |a_n|^2 < +\infty, \tag{26}$$

then

$$|a_n|/n \le \{|a_n|^2 + 1/n^2\}/2;$$

which implies

$$\sum_n |a_n|/n < +\infty. \tag{27}$$

Theorem 10. *If ϕ is L–periodic continuous, with ϕ' piecewise continuous, then the Fourier series of ϕ converges absolutely uniformly to ϕ. Also, the Fourier coefficients of ϕ' are given by*

$$c_n(\phi') = 2\pi i n c_n(\phi)/L. \tag{28}$$

More generally, if ϕ *is* L*–periodic continuous* C^k, $k \geq 0$, *with piecewise continuous* $\phi^{(k+1)}$, *then*

$$c_n(\phi^{(\ell)}) = (2\pi in/L)^\ell c_n(\phi)$$

for all $\ell = 1, 2 \ldots k+1$, *and*

$$n^{k+1} c_n(\phi) \to 0 \tag{29}$$

as $n \uparrow +\infty$. *Furthermore, for every* $j = 1 \ldots k$, *the Fourier series of* $\phi^{(j)}$ *converges absolutely uniformly to* $\phi^{(j)}$.

Proof. If ϕ is L–periodic continuous, and ϕ' is piecewise continuous, then

$$\begin{aligned} c_n(\phi') &= \frac{1}{L}\int_0^L \phi'(x) e^{-2\pi inx/L}\, dx \\ &= \frac{1}{L}\left\{ \phi(x)e^{-2\pi inx/L}\Big|_0^L + \frac{2\pi in}{L}\int_0^L \phi(x)e^{-2\pi inx/L}\, dx \right\} \\ &= \frac{2\pi in c_n(\phi)}{L}, \end{aligned}$$

that is, (28).

Now $c_n(\phi') \to 0$ by the Bessel inequality for ϕ', which implies $nc_n(\phi) \to 0$, the limit (29) with $k = 0$.

It remains to show the uniform absolute convergence of $s_n(\ ;\phi)$ to ϕ. Apply the implication of (27) from (26) to

$$a_n = c_n(\phi'),$$

for ϕ an L–periodic continuous function, with piecewise continuous ϕ'. Then (15), (28), and (27) imply

$$\sum_{n=-\infty}^{+\infty} |c_n(\phi)| < +\infty,$$

which implies, by the Weierstrass M–test, that the Fourier series of ϕ converges absolutely uniformly to a continuous function, which must be ϕ by Theorem 9. We therefore have the first statement of Theorem 10.

Repeated application of this argument yields the second general statement of the theorem.
qed

Theorem 11. *If the Fourier series* (20) *satisfies*

$$|a_n| \leq \frac{\text{const.}}{|n|^{k+1+\epsilon}}, \tag{30}$$

for a given positive integer k, and for all integers n, then it converges absolutely uniformly to a C^k function ϕ. The k derivatives of ϕ may be obtained through term-by-term differentiation of (20)*, and in each case, the new Fourier series converges absolutely uniformly to the corresponding derivative.*

Proof. We are now given the series (20). If

$$|a_n| \leq \text{const.}/|n|^{1+\epsilon}$$

for all n, (that is, $k = 0$ in (21)), for some $\epsilon > 0$, then

$$\sum_n |a_n| < +\infty,$$

which implies by the Weierstrass M–test that the Fourier series (20) converges absolutely uniformly to a continuous function $\phi(x)$ with

$$a_n = c_n(\phi).$$

What if we are given the estimate (20) with $k = 1$, namely,

$$|a_n| \leq \text{const.}/|n|^{2+\epsilon}$$

for all n, for some $\epsilon > 0$? Then the Fourier series (20) converges to some continuous ϕ. Consider the Fourier series

$$\sum_{n=+\infty}^{+\infty} (2\pi in/L)a_n e^{2\pi inx/L}; \tag{31}$$

then

$$|(2\pi in/L)a_n| \leq \text{const.}/|n|^{1+\epsilon}$$

for all n, which implies that the Fourier series (31) converges absolutely uniformly to some continuous ψ. We want to show that

$$\psi = \phi'. \tag{32}$$

How to do it? Set

$$\begin{aligned} \Psi(x) &:= \int_0^x \psi(\xi)\, d\xi \\ &= \int_0^x \sum_{n=-\infty}^{+\infty} (2\pi in/L)a_n e^{2\pi in\xi/L}\, d\xi \end{aligned}$$

$$\begin{aligned} &= \sum_{n=-\infty}^{+\infty} \int_0^x (2\pi in/L) a_n e^{2\pi in\xi/L}\, d\xi \\ &= \sum_{n=-\infty}^{+\infty} a_n \{e^{2\pi inx/L} - 1\} \\ &= \sum_{n=-\infty}^{+\infty} a_n e^{2\pi inx/L} - \sum_{n=-\infty}^{+\infty} a_n \\ &= \phi(x) - \phi(0). \end{aligned}$$

So Ψ differs from ϕ by a constant. This implies (32). One can now repeat the above argument for general $k \geq 2$. qed

§4. Pointwise convergence of $s_N(x;\phi)$ to $\phi(x)$

We assume, for our convenience, that $L = 2\pi$. Then by (4) we have

$$s_N(x;\phi) = \int_{-\pi}^{\pi} D_N(\xi - x)\phi(\xi)\, d\xi;$$

and by (2) we have

$$s_N(x;\phi) = (1/2\pi) \int_{-\pi}^{\pi} \phi(\xi + x) \frac{\sin(N + 1/2)\xi}{\sin \xi/2}\, d\xi. \tag{33}$$

Therefore, to say that $s_N(x;\phi) \to \phi(x)$ is to say D_N converges to a delta function as $N \uparrow +\infty$. But, as we mentioned earlier (example III.4), D_N cannot be an approximation to the identity in the usual sense, because of its oscillatory behaviour. So we must, again, work through the details of the limit argument. First we state the results, and then we discuss the proofs.

Theorem 12. The localization principle. *Given x, the convergence of $s_N(x;\phi)$, as $N \uparrow +\infty$, is completely determined by the restriction of ϕ to any neighborhood of x.*

Theorem 13. *Given x. If ϕ' is piecewise continuous on a neighborhood of x, then*

$$\lim_{N\uparrow+\infty} s_N(x;\phi) = (1/2)\{\phi(x+) + \phi(x-)\}. \tag{34}$$

In particular, if ϕ is continuous at x, and ϕ' is piecewise continuous on a neighborhood of x of x, then $s_N(x;\phi) \to \phi(x)$ as $N \to +\infty$.

Proof of Theorems 12and 13. In (33) write the integral for $s_N(x;\phi)$ as

$$s_N(x;\phi) = (1/2\pi)\int_{-\pi}^{\pi} = (1/2\pi)\int_{|\xi|\le\delta} + (1/2\pi)\int_{\delta\le|\xi|\le\pi}$$

where δ is any chosen number in $(0,\pi)$. The localization principle is then saying that

$$\lim_{N\uparrow+\infty}\frac{1}{2\pi}\int_{\delta\le|\xi|\le\pi}\phi(\xi+x)\frac{\sin(N+1/2)\xi}{\sin\xi/2}\,d\xi = 0 \tag{35}$$

— no matter how we pick δ in $(0,\pi)$. (This is the condition (2) in Theorem III.1.)

To prove (35), write

$$\begin{aligned}\frac{\sin(N+1/2)\xi}{\sin\xi/2} &= \frac{e^{iN\xi}e^{i\xi/2}-e^{-iN\xi}e^{-i\xi/2}}{2i\sin\xi/2}\\ &= \frac{e^{i\xi/2}}{2i\sin\xi/2}e^{iN\xi} - \frac{e^{-i\xi/2}}{2i\sin\xi/2}e^{-iN\xi},\end{aligned}$$

that is,

$$\frac{\sin(N+1/2)\xi}{\sin\xi/2} = \frac{e^{i\xi/2}}{2i\sin\xi/2}e^{iN\xi} - \frac{e^{-i\xi/2}}{2i\sin\xi/2}e^{-iN\xi}.$$

We wish to substitute this expression into the integral in (35). Note that the integral is only over the two intervals $\delta\le|\xi|\le\pi$. Replace $\phi(\xi+x)$ in the integral by the function $\psi(\xi)$ which (i) is identically equal to 0 for $|\xi|\le\delta$, (ii) is equal to $\phi(x+\xi)$ for $\delta\le|\xi|\le\pi$, and (iii) is 2π–periodic. Then

$$\begin{aligned}\frac{1}{2\pi}\int_{\delta\le|\xi|\le\pi}\phi(x+\xi)\frac{\sin(N+1/2)\xi}{\sin\xi/2}\,d\xi &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi(\xi)\frac{\sin(N+1/2)\xi}{\sin\xi/2}\,d\xi\\ &= c_{-N}\left(\frac{\psi(\xi)e^{i\xi/2}}{2i\sin\xi/2}\right) - c_N\left(\frac{\psi(\xi)e^{-i\xi/2}}{2i\sin\xi/2}\right)\\ &\to 0\end{aligned}$$

as $N\uparrow+\infty$ (c_{-N} and c_N denote the respective Fourier coefficients of the indicated functions), since the functions

$$\frac{\psi(\xi)e^{i\xi/2}}{2i\sin\xi/2},\qquad -\frac{\psi(\xi)e^{-i\xi/2}}{2i\sin\xi/2} \tag{36}$$

are piecewise continuous — the only potential trouble spot is at $\xi=0$ (because of $\sin\xi/2$ in the denominator) but $\psi=0$ on a full neighborhood of $\xi=0$. So the functions (36) are piecewise continuous, and (35) is proven.

To prove Theorem 13, we first consider the case when ϕ is actually continuous at x. Of course

$$\int_{-\pi}^{\pi} D_N(\xi)\,d\xi = 1.$$

Therefore, as above,

$$\begin{aligned} s_N(x;\phi) - \phi(x) &= (1/2\pi)\int_{-\pi}^{\pi} \frac{\sin(N+1/2)\xi}{\sin\xi/2}\{\phi(\xi+x)-\phi(x)\}\,d\xi \\ &= (1/2\pi)\int_{-\pi}^{\pi} \frac{\{\phi(\xi+x)-\phi(x)\}e^{i\xi/2}}{2i\sin\xi/2}e^{iN\xi}\,d\xi \\ &\quad +(1/2\pi)\int_{-\pi}^{\pi} \frac{\{\phi(\xi+x)-\phi(x)\}e^{-i\xi/2}}{2i\sin\xi/2}e^{-iN\xi}\,d\xi. \end{aligned}$$

Assume $\{\phi(\xi+x)-\phi(x)\}/\sin\xi/2$ is a piecewise continuous 2π–periodic function of ξ. (That our hypothesis guarantees this fact — we'll see shortly.) Then, as above,

$$\begin{aligned} s_N(x;\phi) - \phi(x) &= c_{-N}\left(\frac{\{\phi(\xi+x)-\phi(x)\}e^{i\xi/2}}{2i\sin\xi/2}\right) - c_N\left(\frac{\{\phi(\xi+x)-\phi(x)\}e^{-i\xi/2}}{2i\sin\xi/2}\right) \\ &\to 0. \end{aligned}$$

What does it take to ensure that $\{\phi(\xi+x\}-\phi(x)\}/\sin\xi/2$ is a piecewise continuous 2π–periodic function of ξ? The only place for which any consideration is necessary is at $\xi = 0$, where we must consider

$$\frac{\phi(\xi+x)-\phi(x)}{\sin\xi/2} = \frac{\phi(\xi+x)-\phi(x)}{\xi}\frac{\xi}{\sin\xi/2}.$$

Of course,

$$\frac{\xi}{\sin\xi/2} \sim 2$$

as $\xi \to 0$.

Therefore, if ϕ is continuous at x, and ϕ' has both right and left sided derivatives at x, then, by the mean value theorem of differential calculus,

$$\{\phi(\xi+x)-\phi(x)\}/\sin\xi/2$$

is a piecewise continuous 2π–periodic function of ξ. By the previous discussion we have $s_N(x;\phi) \to \phi(x)$ as $N \uparrow +\infty$.

What if we are only given that ϕ and ϕ' have right and left sided limits at x? Then we adjust the above argument along the lines of Section II.3 (since $(\sin(N+1/2)\xi)/\sin\xi/2$ is an even function of ξ) to obtain (28). qed

Exercises

Exercise 17. Use Exercise 6 to show that

$$\begin{aligned}\frac{1}{\sin z} &= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z - n\pi} + \frac{1}{z + n\pi}\right], \\ \cot z &= \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{1}{z - n\pi} + \frac{1}{z + n\pi}\right],\end{aligned}$$

where z is any number which is not a multiple of π.

Exercise 18. Sum the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}; \qquad \sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n};$$
$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}; \qquad \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}.$$

Exercise 19. Find the sum of each of the following numerical series by evaluating — at a suitable point — a Fourier series discussed in the problems:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}; \qquad \frac{1}{2a} + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2 + a^2}.$$

Exercise 20. Gibbs' phenomenon. Consider the function

$$S(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases},$$

So $S(x)$ has a jump disconuity at the origin. Then

$$s_N(x) = \sum_{k=-N}^{N} c_k e^{ikx},$$

where c_k denotes the k–th Fourier coefficients of S. Of course, $s_N(x)$ is given by

$$s_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin{(N + 1/2)\xi}}{\sin \xi/2} S(x + \xi)\, d\xi.$$

Now Theorem 10 states that $s_N(0) \to 0$ as $N \uparrow \infty$. The question we ask now is about the approximation of $S(x)$ by $s_N(x)$ near $x = 0$ for large N. We already know what happens if we fix x. When $x \neq 0$

then $s_N(x) \to S(x)$. But what if we fix N? Is it true that if we fix N sufficiently large then $s_N(x)$ will reasonably approximate $S(x)$ for *all* $x \neq 0$? The answer is "no". We sketch below the argument.

(a) Show that $s_N(0) = 0$. Give two arguments.

(b) Show that for $x > 0$,

$$s_N(x) = \frac{1}{\pi}\left\{\int_0^x - \int_{\pi-x}^{\pi}\right\} \frac{\sin{(N+1/2)\xi}}{\sin{\xi/2}}\,d\xi.$$

(c) Use integration-by-parts to show that

$$\left|\int_{\pi-x}^{\pi} \frac{\sin{(N+1/2)\xi}}{\sin{\xi/2}}\,d\xi\right| \leq \text{const.}/N,$$

which goes to 0 as $N \uparrow +\infty$, uniformly in x bounded away from π.

(d) Show that for $0 < x < \pi$ we have

$$\lim_{N\uparrow+\infty} \frac{1}{\pi}\int_0^x \frac{\sin{(N+1/2)\xi}}{\sin{\xi/2}}\,d\xi = 1.$$

(e) Show that for *small* $x > 0$ we may replace the study of

$$\frac{1}{\pi}\int_0^x \frac{\sin{(N+1/2)\xi}}{\sin{\xi/2}}\,d\xi$$

by the study of

$$\frac{1}{\pi}\int_0^x \frac{\sin{(N+1/2)\xi}}{\xi/2}\,d\xi,$$

uniformly with respect to N.

(f) Prove:

$$\lim_{N\uparrow+\infty} \frac{1}{\pi}\int_0^x \frac{\sin{(N+1/2)\xi}}{\xi/2}\,d\xi = 1.$$

(g) Prove:

$$\frac{1}{\pi}\int_0^x \frac{\sin{(N+1/2)\xi}}{\xi/2}\,d\xi = \frac{2}{\pi}\int_0^{(N+1/2)x} \frac{\sin{\xi}}{\xi}\,d\xi.$$

(h) Prove:

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin{\xi}}{\xi}\,d\xi = 1.$$

HINT: Consider the Laplace transform of

$$F(t) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin{t\xi}}{\xi}\,d\xi.$$

(i) Prove:

$$s_N\left(\frac{\pi}{N+1/2}\right) \sim \frac{2}{\pi}\int_0^{\pi} \frac{\sin{\xi}}{\xi}\,d\xi > 1,$$

as $N \uparrow +\infty$. So for every N, no matter how large, one can find points $x > 0$ close to 0 for which $S_N(x)$ stays away from $S(x) = 1$ by a uniform amount.