I. Exponential Functions

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Contents

We consider here, for purposes of review, the definition of the exponential functions in the setting of complex numbers. We do not include lots of detail here (that means this is not a spectator sport); rather we present some old material in a different way in order to emphasize ideas and calculations that will constantly appear in all that follows. Some proofs are included, because they shed light on the ideas. Others might be included because they are extended exercises in calculation. When overwhelming, simply skip them on the first reading.

§1. Complex numbers

Complex numbers are expressions of the form

$$z = x + iy$$

where x and y are real numbers. z is the *complex number*, x its *real part*, and y its *imaginary part*.

Addition subtraction, multiplication and division

If $z = x + iy$ and $w = u + iv$, then we define the *addition of z and w, $z + w$*, by

$$z + w := (x + u) + i(y + v).$$

For complex z, z_1, z_2, z_3 one always has

$$\begin{aligned} (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3), \\ z_1 + z_2 &= z_2 + z_1, \\ z + (0 + i0) &= z. \end{aligned}$$

So $0 + i0$ will be "zero" of complex numbers, written simply as 0.

To subtract the complex number a from the complex number b is, by definition, to solve the equation

$$b = a + z.$$

One always has one and only one solution, which is easily calculated.

Given z and w above, we define *multiplication of z and w, zw,* by

$$zw := (xu - yv) + i(xv + yu).$$

Then for complex z, z_1, z_2, z_3 one always has

$$\begin{aligned} (z_1 z_2) z_3 &= z_1 (z_2 z_3), \\ z_1 z_2 &= z_2 z_1, \\ z(1 + i0) &= z. \end{aligned}$$

So $1 + i0$ is the "one" of the complex numbers.

To divide the complex number b by the complex number a, is, by definition, to solve the equation

$$b = az.$$

So, if

$$a = \alpha + i\beta, \qquad b = \gamma + i\delta,$$

then one wishes to solve

$$\gamma + i\delta = (\alpha + i\beta)(x + iy),$$

that is, to solve the system of 2 equations in 2 unknowns, given by

$$\begin{aligned} \gamma &= \alpha x - \beta y \\ \delta &= \beta x + \alpha y, \end{aligned}$$

where α, β, γ, δ are given, and x and y are to be found. This system has a unique solution if and only if

$$\alpha^2 + \beta^2 > 0,$$

that is, $a \neq 0$, in which case

$$x = \frac{\alpha\gamma + \beta\delta}{\alpha^2 + \beta^2}, \qquad y = \frac{\alpha\delta - \beta\gamma}{\alpha^2 + \beta^2}. \tag{1}$$

In this case we write: $z = b/a$.

One has the usual rules:

$$\begin{aligned} w(z_1 + z_2) &= wz_1 + wz_2, \\ \frac{b}{a} + \frac{d}{c} &= \frac{bc + ad}{ac}, \end{aligned}$$

$$\frac{b}{a} \cdot \frac{d}{c} = \frac{bd}{ac}, \qquad \frac{b}{a} \Big/ \frac{d}{c} = \frac{bc}{ad}.$$

Note the subcollection $\{x + i0\}$ in the complex numbers is actually the collection of real numbers in disguise. By that we mean that if the real numbers x, y are to correspond to the complex numbers $x + i0$, $y + i0$, then we can either add and multiply x with y with the usual rules — inside the real numbers — or we may correspondingly add and multiply $x + i0$ with $y + i0$ — inside the complex numbers. In the second case, the rules above will yield $(x+y)+i0$ for addition, and $xy+i0$ for multiplication. Since these answers correspond to $x+y$ and xy respectively, it makes no difference how one *now* views the real numbers, whether as a stand-alone or whether as part of the complex numbers.

Complex conjugates and absolute values

Given $z = x+iy$, set

$$\overline{z} = x - iy, \quad |z| = \sqrt{x^2+y^2}.$$

We say $\overline{z}$ is the *conjugate of* z, and $|z|$ is the *absolute value of* z.

Then one easily verifies

$$|z|^2 = z\overline{z}, \qquad 1 = \frac{z\overline{z}}{|z|^2} = z\frac{\overline{z}}{|z|^2}, \qquad \frac{1}{z} = \frac{\overline{z}}{|z|^2},$$

which implies a very helpful shortcut for division

$$\frac{b}{a} = b\frac{1}{a} = \frac{b\overline{a}}{|a|^2}.$$

One also has:

$$\overline{z+w} = \overline{z} + \overline{w}, \qquad \overline{zw} = \overline{z}\cdot\overline{w},$$
$$|zw| = |z|\cdot|w|, \qquad |z/w| = |z|/|w|.$$

Furthermore,

$$x = \operatorname{Re} z = \frac{z+\overline{z}}{2}, \qquad y = \operatorname{Im} z = \frac{z-\overline{z}}{2i}.$$

§2. The geometry of complex numbers

To each $z = x+iy$ associate the point (x,y) in the plane.

Note that the mapping $z \to \overline{z}$ corresponds to reflection in the real axis.

For z_1, z_2 one may think of $z_2 - z_1$ as the vector from z_1 to z_2. Therefore,

$$|z_1 - z_2| = d(z_1, z_2),$$

where d denotes the distance in the plane.

From elementary geometric considerations one has the **triangle inequality**

$$|z+w| \le |z| + |w|,$$

from which one easily obtains

$$|z-w| \ge ||z| - |w||.$$

On the other hand, one can also give an algebraic proof of the triangle inequality:

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z}+\overline{w}) \\ &= |z|^2 + 2\mathrm{Re}\, z\overline{w} + |w|^2 \\ &\le |z|^2 + 2|z||w| + |w|^2 \\ &= \{|z|+|w|\}^2. \end{aligned}$$

Now take the square root of both sides of the inequality.

Polar coordinates in the plane

Before introducing polar coordinates in the plane, note that if α is a real number, and

$$z = \cos\alpha + i\sin\alpha,$$

then

$$|z| = 1, \qquad \overline{z} = \cos\alpha - i\sin\alpha = 1/z,$$

and, for α, β real, we have, by the addition formulae for sine and cosine,

$$\{\cos\alpha + i\sin\alpha\}\{\cos\beta + i\sin\beta\} = \cos(\alpha+\beta) + i\sin(\alpha+\beta). \tag{2}$$

In particular, for any integer n, one has

$$\{\cos\alpha + i\sin\alpha\}^n = \cos n\alpha + i\sin n\alpha. \tag{3}$$

Now, given any complex number $z = x + iy$ one determines the *polar coordinates of* z by

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{4}$$

where $r = |z|$, and θ is only defined up to an integral multiple of 2π. We refer to any θ satisfying (4) as an *argument of* z and denote it by $\arg z$; we refer to the *unique* θ in the interval $(-\pi, \pi]$ satisfying (4) as the *principal argument of* z, and denote it by $\mathrm{Arg}\, z$.

From (2) one immediately concludes that if we represent two complex numbers z and w by

$$\begin{aligned} z &= r\{\cos\theta + i\sin\theta\}, \\ w &= \rho\{\cos\phi + i\sin\phi\}, \end{aligned}$$

then

$$zw = r\rho\{\cos(\theta+\phi) + i\sin(\theta+\phi)\},$$

by the addition formulae for sine and cosine, and

$$z^n = r^n\{\cos n\theta + i\sin n\theta\}$$

for all integers n.

Now suppose we are *given* w, and one wishes to *solve* the equation

$$z^n = w, \tag{5}$$

that is, to calculate the n–th root of w. Represent z and w in polar coordinates, as above. Then the equation (5) is equivalent to

$$\rho = r^n, \qquad \phi + 2\pi k = n\theta,$$

where k is *any* integer. We obtain

$$r = \rho^{1/n}, \qquad \theta = \frac{\phi}{n} + \frac{2\pi k}{n},$$

where k ranges over *all* integers. It is important to realize that there are infinitely many solutions for θ, but that there are precisely n solutions for z. Thus every complex number has precisely n distinct n–th roots.

Exercises

Exercise 1. Just checking! Calculate

(a) $(4-3i)+(8+2i)$.

(b) $(3-12i)-(6+2i)$.

(c) $(7+2i)(5-8i)$.

(d) $\frac{8+3i}{4-2i}$.

Exercise 2. Describe geometrically the set of points determined by the following relations:

(a) $|z-2|<5$.

(b) $|z+4i|\geq 3$.

(c) $|z+3i|=|z-4+2i|$.

Exercise 3. Calculate, in form $a+ib$,

(a) $(1+i)^2$.

(b) $(1+\sqrt{3}i)^{23}$.

(c) $(-16)^{1/4}$.

Exercise 4. Solve the equation

$$\frac{x-iy}{x+iy}=i.$$

§3. Two important formulae

The following are used constantly.

Theorem 1. (Binomial theorem) *For all complex z, w, and nonnegative integers n, we have*

$$(z+w)^n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}z^k w^{n-k}. \tag{6}$$

Theorem 2. (Geometric progression) *For all complex $z\neq 1$, and all nonnegative integers n, we have*

$$1+z+\cdots+z^n=\frac{1-z^{n+1}}{1-z}. \tag{7}$$

Exercises

Exercise 5. Show that if $z\neq 1$ is a solution of $z^n=1$, that is, z is a nontrivial n–th root of unity, then

$$1+z+\cdots+z^{n-1}=0.$$

§4. Review of sequences and series

Basic facts for sequences

We assume the reader is comfortable with the notion of a limit of a *sequence* of real numbers.

Definition. We say that a sequence $z_n = x_n + iy_n$ *converges to the complex number* $a = b + ic$ if and only if

$$\lim_{n\uparrow+\infty} x_n = b, \qquad \text{and} \qquad \lim_{n\uparrow+\infty} y_n = c.$$

So the notion of limits of complex sequences reduces to that of real sequences.

If one changes a *finite* number of terms in the sequence, then that does not affect whether the sequence converges or diverges. When the sequence converges, a change of a *finite* number of terms does not affect the actual *value* of the limit. (These facts are simple enough, although they are usually stated by the instructor and then forgotten by the students. But they are used nearly everywhere, even when they are not mentioned explicitly.)

If the sequence z_n converges to z as $n \uparrow +\infty$, then the sequence is *bounded,* that is, there exists a positive number M so that $|z_n| \le M$ for all n.

It is certainly possible that a sequence be bounded, but that it not converge. For example, consider the sequence $z_n = (-1)^n$.

However, suppose that a_n is a strictly increasing sequence of real numbers. Then the sequence a_n converges if and only if it is bounded.

For any $|z| < 1$, one has

$$\lim_{n\uparrow+\infty} z^n = 0. \tag{8}$$

Series

Definition. Start with a *sequence* of complex numbers a_n and *then* consider the new *sequence*

s_n *of partial sums* given by

$$s_n = a_1 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

If the sequence s_n converges to the complex number s, then we say that the *infinite series* $\sum a_n$ *converges to* s. We write the result as

$$\sum_{n=1}^{\infty} a_n = s.$$

Suppose one changes a *finite* number of terms of the *sequence* a_n. Then the fact of convergence or divergence of the *series* $\sum_n a_n$ is unaffected. *However*, if the original series converged, its new limit might be different from that of the original series.

It is standard that if the series converges, that is, if there exists s such that $s_n \to s$, then the individual terms must tend to 0, that is, $a_n \to 0$.

Series of positive terms

Suppose all a_n are *real* and *positive.* Then the sequence of partial sums s_n is strictly increasing. Therefore the series $\sum a_n$ converges if and only if the partial sums s_n are bounded.

The simplest (and most powerful) application of this fact is the **comparison test.** That is, suppose one is given two *real* sequences a_n and b_n such that

$$0 < a_n \leq b_n \tag{9}$$

for all n. The result is that, if the series $\sum b_n$ converges, then the series $\sum a_n$ also converges.

The proof is easy. Let

$$s_n = a_1 + \cdots + a_n, \qquad t_n = b_1 + \cdots + b_n$$

denote the partial sums of the series $\sum a_n$, $\sum b_n$, respectively. If t denotes the infinite sum $\sum b_n$, then $t_n \leq t$ for all n, which implies that $s_n \leq t_n \leq t$ for all n, that is, the strictly increasing sequence s_n is bounded, which implies it converges. qed

The comparison test is rarely used in the precise way it is stated.

First, the comparison (9) need not be valid for all n, it might be allowed to fail for a *finite* number of n.

Second, if there exists a constant c such that

$$0 < a_n \leq cb_n$$

for all $n \geq N$, N a fixed positive integer, then the result of the comparison test would still be valid.

Third, a frequently useful way to state the comparison test is as follows:

Theorem 3. *If the limit* $\lim a_n/b_n$ *exists and is positive, then the series* $\sum a_n$ *converges if and only if the series* $\sum b_n$ *converges.*

Proof. Suppose

$$\lim_n \frac{a_n}{b_n} = L > 0.$$

Then there exists a positive integer N such that

$$\frac{L}{2} \leq \frac{a_n}{b_n} \leq 2L$$

for all $n \geq N$. Suppose the series $\sum b_n$ converges. Then use the right hand side of the inequality, namely

$$a_n \leq 2Lb_n$$

for all $n \geq N$. Since $\sum b_n$ converges, this implies a_n converges. Now suppose that $\sum a_n$ converges. Use the left hand side of the inequality, namely,

$$b_n \leq \frac{2}{L} a_n$$

for all $n \geq N$. Since $\sum a_n$ converges, this implies $\sum b_n$ converges. qed

Example 1. A simple illustration goes as follows: Suppose one wants to prove the convergence of the series

$$\sum_{n=4}^{\infty} \frac{1}{n^2-1}.$$

The natural thing to do would be to compare it to the series

$$\sum_{n=4}^{\infty} \frac{1}{n^2}.$$

But it is not true that

$$\frac{1}{n^2-1} \le \frac{1}{n^2}$$

for large n. In fact, the opposite is true for all n. So one cannot use the first formulation of the comparison test to verify the convergence of the series $\sum 1/(n^2-1)$. Nevertheless, this does not seems to be an important point. Since 1 is insignificant when compared to n^2 for large n, it seems only right that the 1 does not play a significant role in deciding whether the series converges or not. But with the last formulation of the comparison test, the matter is quite easy, since

$$\lim_{n\to\infty} \frac{1}{n^2-1} \Bigg/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1.$$

So the series $\sum 1/(n^2-1)$ converges.

Geometric series and the ratio test

Note that (8) and Theorem 2 above imply for the **geometric series:**

Theorem 4.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \tag{10}$$

whenever $|z| < 1$. If $|z| \ge 1$, then the individual terms z^n do not tend to 0; therefore the series does not converge.

The ratio test is an application of the comparison test and the geometric series. The argument goes as follows.

Theorem 5. *Suppose we are given a sequence a_n of positive real numbers, satisfying*

$$\lim_{n\uparrow+\infty} \frac{a_{n+1}}{a_n} = r < 1.$$

Then the series $\sum a_n$ converges.

Proof. Certainly, for the average ρ of r and 1, that is $\rho = (1/2)(r+1)$, one has the existence of some positive integer N for which

$$\frac{a_{n+1}}{a_n} < \rho < 1$$

for all $n \geq N$. Then

$$a_{N+1} < \rho a_N, \qquad a_{N+2} < \rho a_{N+1} < \rho^2 a_N,$$

and so forth. The general formula now becomes

$$a_{N+k} < \rho^k a_N$$

for all integers $k \geq 0$. Then, by (10)

$$\sum_{n=1}^{N+\ell} a_n = \sum_{n=1}^{N-1} a_n + \sum_{k=0}^{\ell} a_{N+k} \leq \sum_{n=1}^{N-1} a_n + \sum_{k=0}^{\ell} \rho^k a_N \leq \sum_{n=1}^{N-1} a_n + a_N \frac{1}{1-\rho},$$

which shows that the partial sums of $\sum a_n$ are bounded. Therefore $\sum a_n$ converges. qed

Note that argument fails if $r = 1$; so one does not really know from these considerations whether the series converges or diverges.

If $r > 1$ then the series diverges, since the individual terms a_n will not tend to 0, as $n \uparrow +\infty$.

Series with arbitrary terms

We now return to series with arbitrary terms, even complex.

Given any such series $\sum a_n$, we first consider the series $\sum |a_n|$ consisting of the absolute values of the terms of the original series.

If the series $\sum |a_n|$ converges we refer to the original series $\sum a_n$ as *absolutely convergent.*

If the series $\sum |a_n|$ diverges but the original series $\sum a_n$ converges, then we refer to the original series as *conditionally convergent.*

The main theorem here is

Theorem 6. *If the series $\sum a_n$ is absolutely convergent, then it is convergent.*

We give a detailed proof below (see the appendix §8). More detail than one might desire. For we shall characterize precisely when a series is divergent, conditionally convergent, or absolutely convergent. The reader might skip the discussion for the time being, but the ideas in the proof are important for delta functions later on.

The simplest way to state the basic idea is: When a series is absolutely convergent, it must be that the size (absolute value) of its terms tend to 0 very quickly, that even though we add them up the partial sums nevertheless remain bounded. Therefore even if one considers the original series with the signs + or − along with the appropriate terms, the terms will still be so small that the partial sums s_n change very little when n is large. The introduction of the sign can only help matters by introducing cancellations.

One might summarize the matter suggestively by stating that absolute convergence reflects the *damping* of the terms, and conditional convergence reflects the *cancellation effects* of the terms, only, without the damping.

Example 2. Recall the simplest example. Let $a_n = (-1)^{n-1}/n$. Then

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

is conditionally convergent.

To verify that the series is not absolute convergent, one studies $\sum 1/n$, the harmonic series which is (for example, by the integral comparison test) divergent.

The conditional convergence of the series is usually obtained by the Leibniz alternating test, but here one can explicitly show that the original series itself converges. Indeed,

$$\begin{aligned} &1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots + \frac{1}{2n-1} - \frac{1}{2n} + - \cdots \\ &= \left\{1 - \frac{1}{2}\right\} + \left\{\frac{1}{3} - \frac{1}{4}\right\} + - \cdots + \left\{\frac{1}{2n-1} - \frac{1}{2n}\right\} + - \cdots \\ &= \frac{1}{2} + \frac{1}{12} + \cdots + \frac{1}{(2n)(2n-1)} + \cdots, \end{aligned}$$

which converges by comparison to the series $\sum 1/4n^2$. So it is the cancellation that does the work.

We now discuss the addition and multiplication of series. Certainly, if both $\sum a_n$ and $\sum b_n$ converge, then one has

$$\sum_n \{a_n + b_n\} = \sum_n a_n + \sum_n b_n,$$

that is, the series whose indivdual terms are $a_n + b_n$ converges, and its sum is the sum of the separate series. For multiplication one has a more complicated expression. The formal theorem is as follows:

Theorem 7. *Assume the convergence of the series $\sum a_n$ and $\sum b_n$, and that at least one of them is absolutely convergent. Then the series $\sum_n \sum_{k=0}^n a_k b_{n-k}$ converges and*

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} a_k b_{n-k} \right\} = \left\{ \sum_{n=0}^{\infty} a_n \right\} \left\{ \sum_{n=0}^{\infty} b_n \right\}.$$

Power series

We now discuss *power series.* Namely, we study functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n,$$

where a_n, α are complex numbers, and z ranges over the complex numbers.

Then for the ratio test one studies

$$\frac{|a_{n+1}(z-\alpha)^{n+1}|}{|a_n(z-\alpha)^n|} = \frac{|a_{n+1}|}{|a_n|}|z-\alpha|.$$

Therefore, if the sequence $|a_{n+1}|/|a_n| \to R$ as $n \uparrow +\infty$, one concludes that the series is absolutely convergent inside the *circle* of radius $1/R$, with center α; that is, the series converges absolutely for $|z-\alpha| < 1/R$; that is, the *domain of* f contains the disk $\{z : |z-\alpha| < 1/R\}$. The series certainly diverges for $|z-\alpha| > 1/R$ — so the only other possible points of convergence are precisely on the circle $|z-\alpha| = 1/R$.

The most elementary result on the continuity of power series $\sum a_n(z-\alpha)^n$ is that it is continuous at the origin $z = \alpha$. In particular,

$$\lim_{z\to\alpha} \sum_{n=0}^{\infty} a_n (z-\alpha)^n = a_0.$$

Actually, it is standard that inside the circle of convergence of a power series that f is more than continuous — f is differentiable, in the sense that

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

always exists, and may be calculated by differentiation of the series term-by-term. But if so, then the radius of convergence of the series for f' is the same as that of f; so we may differentiate again. We conclude that f has derivatives of all orders.

In particular, the n–th derivative of $f(z)$ at $z = \alpha$ is given by **Taylor's formula**

$$f^{(n)}(\alpha) = n! a_n. \tag{11}$$

A side-comment is that using power series one can motivate the guess for the formula for the product of series. Assume one is given two power series $\sum_n a_n z^n$, $\sum_n b_n z^n$ with the same disk of convergence (it is centered at the origin). If we were to think of the power series as infinite polynomials, multiply the two and collect all the terms associated to the power z^n, then the resulting coefficient would be $\sum_{k=0}^n a_k b_{n-k}$.

Binomial series

Recall the binomial theorem, that

$$(1+z)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k = \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} z^k,$$

the formula valid for all z , for positive integers k.

Now let α be *any* rational number. In order to calculate the function $\phi(z) = (1+z)^\alpha$ in a neighborhood of $z = 0$, satisfying $\phi(0) = 1$, the Theory of Complex Variables says that the function ϕ has a power series expansion about $z = 0$ with radius of convergence equal to 1. Then Taylor's formula (11) for the coefficients of the series implies that

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} z^k. \tag{12}$$

Of course one can now verify that the radius of convergence is equal to 1.

Exercises

Exercise 6. Check for the convergence or divergence of the following sequences and/or series.

(a) The sequence $a_n = (-1)^n n/(n^2+1)$.

(b) The sequence $\sin \pi n/2$.

(c) The series with $a_n = (-1)^n n/(n^2+1)$.

(d) The sequence $a_n = (4n^3 - 3n^2 + 5n - 1)/(-2n^3 + 10n^2 + 37n + 63)$.

(e) The series with $a_n = 5^{n^2}/10^{n^{5/2}}$.

(f) The series with $a_n = (4n^3 - 3n^2 + 5n - 1)/(-2n^3 + 10n^2 + 37n + 63)$.

Exercise 7. Determine the radius of convergence of the following power series:

$$\textbf{(a)} \qquad \sum_{n=1}^{\infty} (-i)^n (z+2)^n$$

$$\textbf{(b)} \qquad \sum_{n=1}^{\infty} \frac{n-5}{n^2+4} z^n$$

$$\textbf{(c)} \qquad \sum_{n=1}^{\infty} \frac{n^{132}}{n!} (z+4)^n$$

$$\textbf{(d)} \qquad \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} (z-1)^n$$

Exercise 8. Let $P(x)$ and $Q(x)$ be any polynomial functions. Show that the ratio test for the series

$$a_n = \frac{P(n)}{Q(n)}$$

is always inconclusive. Similarly, show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)} (z-a)^n$$

is equal to 1, for any choice of a.

Exercise 9. Prove, for $|z| < 1$, that

$$\frac{1}{(z-1)^2} = \sum_{n=1}^{\infty} n z^{n-1}.$$

There are at least *three* proofs. Give them all.

The first is to start with the geometric series, and then multiply it by itself.

The second is to start with the geometric series and differentiate it term-by-term.

The third is by using Taylor's formula (11).

§5. The exponential function

For any complex z we *define*

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{13}$$

The ratio test shows that the power series converges for all z.

The first job is to identify $E(z)$ as an exponential function, that is, there should exist a complex number c so that for any rational number p/q one has $E(p/q) = c^{p/q}$. The key is the identity

$$E(z+w) = E(z)E(w) \tag{14}$$

for all z and w. The proof of (14) is:

$$E(z)E(w) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{j=0}^{\infty} \frac{w^j}{j!} = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \frac{z^\ell w^{n-\ell}}{\ell!(n-\ell)!} \cdots$$

$$\cdots = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\ell=0}^{n} \frac{n!}{\ell!(n-\ell)!} z^\ell w^{n-\ell} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = E(z+w)$$

— the second equality is the multiplication of power series; and the next to last equality is the binomial theorem.

One immediately has

$$E(z)E(-z) = E(0) = 1 \tag{15}$$

for all z; which implies that $E(z)$ is never equal to 0. By the series expansion, the function $E(z)$ is positive when z is real and positive. Since the function $E(z)$ is never equal to 0, $E(z)$ must be positive for *all real* values of z.

From (15) one immediately has

$$E(-z) = \frac{1}{E(z)}. \tag{16}$$

Also, for any integer k and complex number z, one has from (14)

$$E(kz) = (E(z))^k, \tag{17}$$

which implies, also, that for any fraction p/q and complex number z,

$$E((p/q)z) = (E(z))^{p/q}.$$

This suggests that we *define* the positive real number e by

$$e = E(1) = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

It is then a *fact* that

$$e^{p/q} = E(p/q)$$

for any fraction p/q. It is therefore common to write e^z for $E(z)$ for *all* complex z.

The differentiation of the power series (13) term-by-term implies

$$E'(z) = E(z) \tag{18}$$

for all z.

Now, restrict the values of z to the real axis. Then e^z is always positive for all real values of z. By (18) the derivative must always be positive, therefore $y = e^x$ is always strictly increasing as x varies from $-\infty$ to $+\infty$. From the series one has $e^x > x$ for all *positive* x. One concludes

$$\lim_{x\uparrow+\infty} e^x = +\infty; \tag{19}$$

and (16) and (19) combine to imply

$$\lim_{x\downarrow-\infty} e^x = 0 \tag{20}$$

— so e^x increases from 0 to $+\infty$ as x varies from $-\infty$ to $+\infty$.

One can easily sharpen (19) and (20) to

$$\lim_{x\uparrow+\infty} x^{-\alpha}e^x = +\infty, \tag{21}$$

and

$$\lim_{x\downarrow-\infty} |x|^{\alpha}e^x = 0, \tag{22}$$

for all real constants α.

Return to any complex z. Of course, for $z = x + iy$,

$$e^z = e^{x+iy} = e^x e^{iy}.$$

Now e^x is known, but what about e^{iy}? Well,

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iy)^{2k}}{(2k)!} + \sum_{n=0}^{\infty} \frac{(iy)^{2k+1}}{(2k+1)!} \cdots$$

$$\cdots = \sum_{n=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!} + i\sum_{n=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!} = \cos y + i\sin y,$$

that is,

$$e^{iy} = \cos y + i\sin y. \tag{23}$$

One can recapture — that is, *derive* — the addition formulae (2) for sine and cosine from equations (14) and (23). The argument goes as follows: The addition formula for the exponential function (14) implies

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}. \tag{24}$$

The left hand side of (24) is

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta);$$

and the right hand side is

$$\begin{aligned}&(\cos\,\alpha + i\sin\,\alpha)(\cos\,\beta + i\sin\,\beta)\\ &= (\cos\,\alpha\cos\,\beta - \sin\,\alpha\sin\,\beta) + i(\sin\,\alpha\cos\,\beta + \cos\,\alpha + \sin\,\beta),\end{aligned}$$

which *proves* the formulae

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\,\alpha\cos\,\beta - \sin\,\alpha\sin\,\beta,\\ \sin(\alpha+\beta) &= \sin\,\alpha\cos\,\beta + \cos\,\alpha\sin\,\beta.\end{aligned}$$

In what comes later, there will be more than one occasion for which the complex exponential function will be employed in efficient, elementary computations involving sines and cosines. At any rate, one also has, rather easily,

$$e^z = e^x e^{iy} = e^x\{\cos y + i\sin y\} = e^x\cos y + ie^x\sin y;$$

and

$$|e^z| = e^x, \qquad |e^{iy}| = 1;$$

$$e^{2\pi i} = 1. \tag{25}$$

From (14) and (25) we have, for any integer k,

$$e^{z+2\pi ik} = e^z$$

—*so* e^z *is periodic with period equal to* $2\pi i$.

Finally, one checks that

$$e^{-iy} = \cos y - i\sin y;$$

$$\cos y = \frac{e^{iy}+e^{-iy}}{2}; \qquad \sin y = \frac{e^{iy}-e^{-iy}}{2i}.$$

Exercises

Exercise 10. Estimate the speed with which the series converges, namely, for any given z, and $N > |z|$, show that

$$\left|E(z) - \sum_{n=0}^{N} \frac{z^n}{n!}\right| \leq \frac{|z|^{N+1}}{(N+1)!} \frac{1}{1-(z/N)}.$$

HINT: First check that

$$\left|E(z) - \sum_{n=0}^{N} \frac{z^n}{n!}\right| \leq \sum_{n=N+1}^{\infty} \frac{|z|^n}{n!} = \frac{|z|^{N+1}}{(N+1)!} \left\{1 + \frac{|z|}{N+2} + \frac{|z|^2}{(N+2)(N+3)} + \cdots\right\}.$$

Now check that the bracket is less than the geometric series with ratio $|z|/N$. That should do it.

Exercise 11.

(a) For **Dirichlet's kernel** given by

$$\mathbf{D}_N(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inx},$$

show that

(26) $$\mathbf{D}_N(x) = \frac{1}{2\pi} \frac{\sin(N+1/2)x}{\sin x/2}.$$

(b) Also, prove that

(27) $$1 + \cos x + \cdots + \cos Nx = \frac{1}{2} + \frac{\sin(N+1/2)x}{2\sin x/2},$$

(28) $$\sin x + \cdots + \sin Nx = \frac{\{\sin Nx/2\}\{\sin(N+1)x/2\}}{\sin x/2}.$$

(c) Finally, evaluate the series

$$\sum_{n=0}^{\infty} e^{-nx} \cos ny.$$

HINT: For (a), use the geometric progression formula (with a little calculation) to show

$$\sum_{n=-N}^{N} e^{inx} = \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{ix}}.$$

Then show how to convert this last term to $(\sin{(N+1/2)x})/\sin{x/2}$. The rest should be easy.

Exercise 12. For **Fejer's kernel**

$$\mathbf{F}_N(x) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{D}_n(x),$$

show that

(29) $$\mathbf{F}_N(x) = \frac{1}{2\pi N} \left\{\frac{\sin Nx/2}{\sin x/2}\right\}^2.$$

§6. Complex functions of a real variable

Here t is a real variable and

$$\phi(t) = u(t) + iv(t)$$

a complex-valued function of t.

To speak of limits, we define

$$\lim_{t \to t_0} \phi(t) = \lim_{t \to t_0} u(t) + i \lim_{t \to t_0} v(t).$$

That is, $\phi(t)$ has a limit as $t \to t_0$ precisely when both $u(t)$ and $v(t)$ have limits when $t \to t_0$. When this is, in fact, the case then the limit of $\phi(t)$, as $t \to t_0$, is given by the above form.

Similarly, to consider $\phi'(t)$ one studies the limit

$$\lim_{h \to 0} \frac{\phi(t+h) - \phi(t)}{h};$$

the limit exists if and only if $u'(t)$ and $v'(t)$ exist, in which case

$$\phi'(t) = u'(t) + iv'(t).$$

If ϕ is continuous we may define

$$\int_\alpha^\beta \phi(t)\,dt = \int_\alpha^\beta u(t)\,dt + i \int_\alpha^\beta v(t)\,dt.$$

If z is a complex number, and

$$\phi(t) = e^{zt},$$

then

$$\phi'(t) = ze^{zt}.$$

Exercise

Exercise 13. Find the antiderivatives

$$\int e^{\alpha t} \cos \beta t\,dt, \qquad \int e^{\alpha t} \sin \beta t\,dt$$

for real constants α, β.

§7. The gamma function

The classical Gamma function $\Gamma(x)$ is given by

$$\Gamma(x) := \int_0^{+\infty} e^{-t}t^{x-1}\,dt.$$

The integral converges for $x > 0$, and one verifies that

$$\Gamma(1) = 1,$$

and, by integration-by-parts, that

$$\Gamma(x+1) = \int_0^{\infty} e^{-t}t^{x}\,dt = -\left.\frac{t^x}{e^t}\right|_0^{+\infty} + x\int_0^{\infty} e^{-t}t^{x-1}\,dt = x\Gamma(x).$$

So for positive integers n we have

$$\Gamma(n+1) = n!\,.$$

To evaluate $\Gamma(1/2)$ one has

$$\Gamma(1/2) = \int_0^{\infty} e^{-t}t^{-1/2}\,dt.$$

Note that $t^{-1/2}\,dt$ is essentially the differential of $t^{1/2}$. Therefore make the substitution $t = s^2$ and obtain

$$\Gamma(1/2) = 2\int_0^{\infty} e^{-s^2}\,ds = \int_{\mathbb{R}} e^{-s^2}\,ds.$$

To evaluate *this* integral use the following trick:

$$\left\{\int_{\mathbb{R}} e^{-s^2}\,ds\right\}^2 = \left\{\int_{\mathbb{R}} e^{-s^2}\,ds\right\}\left\{\int_{\mathbb{R}} e^{-\sigma^2}\,d\sigma\right\} = \int_{\mathbb{R}^2} e^{-s^2}e^{-\sigma^2}\,ds\,d\sigma = \int_{\mathbb{R}^2} e^{-|\mathbf{x}|^2}\,dA(\mathbf{x}).$$

Now introduce polar coordinates into the plane. Then

$$\int_{\mathbb{R}^2} e^{-|\mathbf{x}|^2}\,dA(\mathbf{x}) = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2}r\,dr = 2\pi\left\{\left.\frac{e^{-r^2}}{2}\right|_0^{+\infty}\right\} = \pi.$$

So

$$\Gamma(1/2) = \int_{\mathbb{R}} e^{-s^2}\,ds = \sqrt{\pi}.$$

Of course, one now knows the specific values of $\Gamma(x)$ for all positive half-integers.

§8. Appendix: absolute and conditional convergence

So we are given a *real* series $\sum a_n$, for which we wish to study absolute and conditional convergence. The idea is quite simple: first look separately at those terms which are positive, and those which are negative. That is, if $a_n > 0$ let $p_n = a_n$, otherwise, if $a_n < 0$ set $p_n = 0$. Similarly, if $a_n < 0$ let $q_n = |a_n| = -a_n$, otherwise, if $a_n > 0$ set $q_n = 0$. Then, for the partial sums:

$$\begin{aligned} \sum_{n=1}^{N} a_n &= \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n \\ \sum_{n=1}^{N} |a_n| &= \sum_{n=1}^{N} p_n + \sum_{n=1}^{N} q_n. \end{aligned}$$

One therefore distinguishes the following cases:

1. Both series $\sum p_n$ and $\sum q_n$ converge. This corresponds to the case of absolute convergence of $\sum a_n$, which implies convergence of the original series $\sum a_n$.

2. Only one of the series $\sum p_n$, $\sum q_n$ converges. Then we must have the divergence of $\sum a_n$.

3. Both series $\sum p_n$ and $\sum q_n$ diverge. Then it is *possible* that the original series $\sum a_n$ converge, even though it cannot be absolutely convergent.

The case of series of *complex* numbers can be reduced (by considering the real and imaginary parts separately) to the real case discussed above.