II. Delta Functions

Isaac Chavel

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1. Delta functions

It is best to start with a formal statement of main theorem. It is a bit of a mouthful, so we state it patiently, then try to explain what it means, and then sketch the idea of the proof.

We are given a 1–parameter family of *nonnegative* functions

$$K_h : (-\infty, +\infty) \to [0, +\infty), \quad h > 0,$$

where h denotes the parameter. The first property we are given is that for every $h > 0$ we have

$$\int_{-\infty}^{+\infty} K_h(\xi)\, d\xi = 1. \tag{1}$$

Secondly we assume that

$$\lim_{h \downarrow 0} \int_{|\xi| \geq \delta} K_h(\xi)\, d\xi = 0 \tag{2}$$

for any fixed $\delta > 0$; that is, the integral of K_h over the *exterior* of any interval about the origin tends to 0 as $h \downarrow 0$ — so all of the integral is piling up close to 0 as h becomes small.

Theorem 1. *Then for any bounded function $f(\xi)$, which is continuous at x, we have*

$$\lim_{h \downarrow 0} \int_{-\infty}^{+\infty} K_h(\xi - x) f(\xi)\, d\xi = f(x). \tag{3}$$

Remark 1. To appreciate the theorem we discuss the two conditions.

The meaning of (1) is that for every $h > 0$, $K_h(\xi)$ describes an averaging process. For example, if

$$K_h(\xi) = \begin{cases} 1/h & |\xi| < h/2 \\ 0 & |\xi| \geq h/2, \end{cases}$$

then

$$\int_{-\infty}^{+\infty} K_h(\xi - x) f(\xi)\, d\xi = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(\eta)\, d\eta$$

is the average value of f over the interval $[x - h/2,\ x + h/2]$, with the value of f at each point of the interval weighted the same as any other point of the interval. Then (1) generalizes this example in two ways. First it allows the possibility that not all points of the interval, in question, be given equal weight in the averaging procedure, namely, K_h need not be constant; and second, that the interval over which the averaging procedure takes place need not be finite.

The meaning of (2) is that as $h \downarrow 0$, the averaging procedure discards the contribution of f, to the average, from all points away from $\xi = 0$. We emphasize, in the spirit of absolute convergence versus ordinary convergence, that the nonnegativity of K_h means that the limit in (2) is the result of "damping" the function f by K_h, and not by any "cancellation" of the values of f induced by K_h.

The theorem then argues that if f is continuous at x, then $f(\xi)$ is practically equal to $f(x)$ for ξ near x; and since $K_h(\xi - x)$ discards information away from $\xi = x$ as $h \downarrow 0$, the only thing left in the limit is the average of an essentially constant function, that is, the average is equal to the constant $f(x)$.

Here is the calculation and the estimate: First,

$$\int_{-\infty}^{+\infty} K_h(\xi - x) f(\xi)\, d\xi - f(x) = \int_{-\infty}^{+\infty} K_h(\xi - x)\{f(\xi) - f(x)\}\, d\xi = \int_{|\xi - x| \leq \delta} + \int_{|\xi - x| \geq \delta}.$$

To show that this is small we estimate:

$$\begin{aligned} \left| \int_{|\xi-x| \leq \delta} K_h(\xi - x)\{f(\xi) - f(x)\}\, d\xi \right| &\leq \int_{|\xi-x|\leq\delta} K_h(\xi - x)|f(\xi) - f(x)|\, d\xi \\ &\leq \left\{ \sup_{|\xi-x|\leq\delta} |f(\xi) - f(x)| \right\} \int_{|\xi-x|\leq\delta} K_h(\xi - x)\, d\xi \\ &\leq \sup_{|\xi-x|\leq\delta} |f(\xi) - f(x)|; \end{aligned}$$

and

$$\begin{aligned}\left|\int_{|\xi-x|\geq\delta} K_h(\xi-x)\{f(\xi)-f(x)\}\,d\xi\right| &\leq \int_{|\xi-x|\geq\delta} K_h(\xi-x)|f(\xi)-f(x)|\,d\xi \\ &\leq \int_{|\xi-x|\geq\delta} K_h(\xi-x)\{|f(\xi)|+|f(x)|\}\,d\xi \\ &\leq 2\{\sup|f|\}\int_{|\xi-x|\geq\delta} K_h(\xi-x)\,d\xi \\ &= 2\{\sup|f|\}\int_{|\xi|\geq\delta} K_h(\xi)\,d\xi.\end{aligned}$$

Note how the first inequality of each of the estimates uses the fact that K_h is nonnegative. Also note that the first string of inequalities highlights the continuity of f at x, and the second string of inequalities highlights discarding information away from x.

In summary, we have

$$\left|\int_{-\infty}^{+\infty} K_h(\xi-x)f(\xi)\,d\xi - f(x)\right| \leq \sup_{|\xi-x|\leq\delta}|f(\xi)-f(x)| + 2\{\sup|f|\}\int_{|\xi|\geq\delta} K_h(\xi)\,d\xi. \tag{4}$$

As $h \downarrow 0$, the second integral on the right hand side of (4) goes to 0, we are only left with the error measured by the continuity of f at x. But if f is continuous at x, we may make this error as small as we like by fixing δ sufficiently small. This is the argument of the theorem.

Definition. The family $K_h(\xi)$ is referred to as an *approximation to the identity*; and it is common to write

$$\lim_{h\downarrow 0} K_h(\xi-x) = \delta_x(\xi),$$

and to refer to δ_x as *the delta function at* x.

Construction of examples

A very general method of manufacturing approximations to the identity goes as follows: Pick any *nonnegative* $\phi(x)$ satisfying

$$\int_{-\infty}^{+\infty} \phi(\xi) = 1; \tag{5}$$

and set

(6) $$K_h(\xi) = \left(\frac{1}{h}\right)\phi\left(\frac{\xi}{h}\right).$$

Then $K_h(\xi)$ will satisfy (1) and (2) of Theorem 1.

Example 1.

$$\phi(\xi) = \begin{cases} 1 & |\xi| < 1/2 \\ 0 & |\xi| \geq 1/2, \end{cases}$$

Then we recapture the simplest example:

$$K_h(\xi) = \begin{cases} 1/h & |\xi| < h/2 \\ 0 & |\xi| \geq h/2. \end{cases}$$

Example 2. Set

$$\phi(\xi) = \frac{1}{\pi}\frac{1}{1+\xi^2}.$$

Then ϕ satisfies (5); so following the prescription of (6) we have

$$K_h(\xi) = \frac{1}{\pi}\frac{h}{h^2+\xi^2}.$$

Example 3. Set

$$\phi(\xi) = \frac{e^{-\xi^2}}{\sqrt{\pi}},$$

Then ϕ satisfies (5); so following the prescription of (6), where we replace h with $\sqrt{4h}$, we have

$$K_h(\xi) = \frac{e^{-\xi^2/4h}}{\sqrt{4\pi h}}$$

Symmetric approximations to the identity

We give the following refinement of Theorem 1, which allows f to have jump discontinuities.

Theorem 2. *Suppose, in addition to* (1) *and* (2), *the family of functions* $K_h(\xi)$ *satisfies*

$$K_h(\xi) = K_h(-\xi), \tag{7}$$

that is, K_h *is symmetric with respect to* $\xi = 0$. *Then for* f *bounded, with jump discontinuity at* x, *we have*

$$\int_{-\infty}^{+\infty} K_h(\xi - x) f(\xi)\, d\xi \to \frac{f(x+) + f(x-)}{2} \tag{8}$$

as $h \downarrow 0$.

Proof. The basic insight is that (7) implies

$$\int_{-\infty}^{x} K_h(\xi - x)\, d\xi = \int_{x}^{+\infty} K_h(\xi - x)\, d\xi = 1/2$$

for any choice of x. Therefore

$$\begin{aligned}
&\int_{-\infty}^{+\infty} K_h(\xi - x) f(\xi)\, d\xi - (1/2)\{f(x+) + f(x-)\} \\
&= \int_{-\infty}^{x} K_h(\xi - x) f(\xi)\, d\xi - f(x-)/2 \\
&\quad + \int_{x}^{+\infty} K_h(\xi - x) f(\xi)\, d\xi - f(x+)/2 \\
&= \int_{-\infty}^{x} K_h(\xi - x)\{f(\xi) - f(x-)\}\, d\xi \\
&\quad + \int_{x}^{+\infty} K_h(\xi - x)\{f(\xi) - f(x+)\}\, d\xi \\
&= \int_{-\infty}^{x-\delta} K_h(\xi - x)\{f(\xi) - f(x-)\}\, d\xi \\
&\quad + \int_{x-\delta}^{x} K_h(\xi - x)\{f(\xi) - f(x-)\}\, d\xi \\
&\quad + \int_{x}^{x+\delta} K_h(\xi - x)\{f(\xi) - f(x+)\}\, d\xi \\
&\quad + \int_{x+\delta}^{+\infty} K_h(\xi - x)\{f(\xi) - f(x+)\}\, d\xi.
\end{aligned}$$

For $h \downarrow 0$, the first and fourth integrals are small because ξ is "far" from x, and the second and third integrals are small because of the one-sided limits.

§2. Delta functions on the circle

Here we consider periodic approximations to the identity, or, equivalently, approximations to the identity on the circle.

Definition. Given a positive number L, a function $f(x)$ on the real line is said to be *L–periodic* if

$$f(x+L) = f(x)$$

for all x in $(-\infty, +\infty)$. One easily sees that an equivalent way of considering such a function is to view it as a function on a circle of length L.

An *L–periodic approximation to the identity* is a family of *nonnegative* L–periodic functions

$$\mathbf{K}_h : (-\infty, +\infty) \to [0, +\infty), \qquad h > 0$$

with the properties:

(9)
$$\int_{-L/2}^{L/2} \mathbf{K}_h(\xi)\, d\xi = 1$$

for all $h > 0$;

(10)
$$\lim_{h \downarrow 0} \int_{\delta \le |\xi| \le L/2} \mathbf{K}_h(\xi)\, d\xi = 0$$

for any fixed δ in $(0, L/2)$. (Of course, in (9), any interval of length L will do.)

As in Theorem 1, we obtain

(11)
$$\lim_{h \downarrow 0} \int_{-L/2}^{L/2} \mathbf{K}_h(\xi - x) f(\xi)\, d\xi = f(x)$$

for all L–periodic bounded functions $f(x)$ which are continuous at x.

A simple method of construction of L–periodic approximations to the identity goes as follows: Given any approximation to the identity on $(-\infty, +\infty)$, $K_h(\xi)$, then consider the function

$$\mathbf{K}_h(\xi) = \sum_{n=-\infty}^{+\infty} K_h(\xi + nL).$$

Then the family $\mathbf{K}_h$ is the example we are seeking. Certainly, $\mathbf{K}_h$ is L–periodic. Also,

$$\int_{-L/2}^{L/2} \mathbf{K}_h(\xi)\, d\xi \quad = \quad \int_{-L/2}^{L/2} \sum_{n=-\infty}^{+\infty} K_h(\xi + nL)\, d\xi$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{+\infty} \int_{-L/2}^{L/2} K_h(\xi + nL)\, d\xi \\
&= \sum_{n=-\infty}^{+\infty} \int_{(n-1/2)L}^{(n+1/2)L} K_h(\xi)\, d\xi \\
&= \int_{\infty}^{+\infty} K_h(\xi)\, d\xi \\
&= 1.
\end{aligned}
$$

And finally, the same change of variables implies that

$$\int_{\delta\le|\xi|\le L/2} \mathbf{K}_h(\xi)\, d(\xi) \le \int_{\delta\le|\xi|} K_h(\xi)\, d\xi \to 0$$

as $h \downarrow 0$.

Example 1. (Here $N \uparrow +\infty$, instead of $h \downarrow 0$). Consider the Dirichlet kernel (cf. Exercise I.11)

(12) $$\mathbf{D}_N(\xi) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{in\xi}$$

for ξ in $(-\infty, +\infty)$. Then $\mathbf{D}_N(\xi)$ is 2π–periodic, and

(13) $$\mathbf{D}_N(\xi) = \frac{1}{2\pi} \frac{\sin(N + 1/2)\xi}{\sin \xi/2}$$

for $|\xi| \le \pi$, with

(14) $$\int_{-\pi}^{\pi} \mathbf{D}_N(\xi)\, d\xi = 1$$

for all N (by (12)); but $\mathbf{D}_N$ cannot qualify to be an approximation to the identity, as described above, because $\mathbf{D}_N$ does not have constant sign. However, Fejer's kernel (cf. Exercise I.12)

$$\mathbf{F}_N(\xi) = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{D}_n(\xi) = \frac{1}{2\pi N} \left\{ \frac{\sin N\xi/2}{\sin \xi/2} \right\}^2$$

does qualify. Indeed, $\mathbf{F}_N$ is always nonnegative; its integral over $[-\pi, \pi]$ is equal to 1; and

$$\int_{\delta\le|\xi|\le\pi} \mathbf{F}_N(\xi)\, d\xi = \frac{2}{2\pi N} \int_{\delta}^{\pi} \left\{ \frac{\sin N\xi/2}{\sin \xi/2} \right\}^2 d\xi \le \frac{1}{\pi N} \frac{\pi - \delta}{\sin^2 \delta/2} \to 0$$

as $N \uparrow +\infty$.