Corrections and Additions to: “Riemannian Geometry: a modern introduction”

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This is a file of corrections for my book, with added hints and sketches of solutions to problems, and added comments on the text itself. I hope to update this file as I work my way through the book (probably chapter-by-chapter), and incorporate the material here into a new edition of the book. My thanks to friends and reveiwers (especially, Jerry Kazdan and the geometers at the University of Pennsylvania) for keeping me honest and for helpful comments and suggestions. Thank you — the reader — for your interest, your patience in putting up with misprints, and worse. I hope the mathematics I chose to present is its own reward.

Chapter I. Riemannian manifolds

Correction. Page 10, Line 12:
`Reads as:` (1.12) —
`Should read as:` (1.12), (1.13) —

Correction. Page 11, Line -1:
`Reads as:` $\omega_{*|o}\frac{\partial}{\partial t}$,
`Should read as:` $\omega_{*|0}\frac{\partial}{\partial t}$,

Addition. Page 12, Line -5: `replace the paragraph starting with` “One shows ...” `with:` To show that T and R are well-defined (that is, they are independent of the extensions of ξ, η, ζ) it is best to change the point-of-view and consider T and R as defined on the vector fields X, Y, Z. We first show that T and R are multilinear (with respect to X, Y, Z) over functions on M, that is, for example,

$$T(X, Y) = \nabla_Y X - \nabla_X Y - [Y, X]$$

satisfies

$$T\left(\sum_{i=1}^{2} f_i X_i, \sum_{j=1}^{2} g_j Y_j\right) = \sum_{i,j=1}^{2} f_i g_j T(X_i, Y_j).$$

Since $T(X,Y) = -T(Y,X)$ we only have to show that T is linear over functions with respect to the first variable. Clearly $T(X,Y)$ is additive with respect to X, and

$$\begin{aligned} T(fX,Y) &= \nabla_Y(fX) - \nabla_{fX}Y - [Y, fX] \\ &= (Yf)X + f\nabla_Y X - f\nabla_X Y - \{(Yf)X + f[X,Y]\} \\ &= fT(X,Y). \end{aligned}$$

So T is linear in X, hence multilinear in X and Y over functions. But then, in any coordinate chart $x : U \to \mathbb{R}^n$, we have for

$$X = \sum_j \xi^j \frac{\partial}{\partial x^j}, \qquad Y = \sum_k \eta^k \frac{\partial}{\partial x^k},$$

the calculation

$$T(X,Y) = \sum_{j,k} \xi^j \eta^k T\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right).$$

So $T(X,Y)_{|p}$ is determined exclusively by the values of $X_{|p}$ and $Y_{|p}$. The discussion for the derivative tensor field R is similar.

Correction. Page 13, Line 5:
`Reads as:` tensors,
`Should read as:` tensor fields,

Addition. Page 13, Line 11: `Insert after equation (1.21):` To prove (1.21) it suffices to check that in any chart $x : U \to \mathbb{R}^n$ we have

$$R(\partial_j, \partial_k)\partial_l + R(\partial_l, \partial_j)\partial_k + R(\partial_k, \partial_l)\partial_j = 0.$$

Well,

$$\begin{aligned} &R(\partial_j, \partial_k)\partial_l + R(\partial_l, \partial_j)\partial_k + R(\partial_k, \partial_l)\partial_j \\ &= \nabla_{\partial_k}\nabla_{\partial_j}\partial_l - \nabla_{\partial_j}\nabla_{\partial_k}\partial_l + \nabla_{\partial_j}\nabla_{\partial_l}\partial_k - \nabla_{\partial_l}\nabla_{\partial_j}\partial_k + \nabla_{\partial_l}\nabla_{\partial_k}\partial_j - \nabla_{\partial_k}\nabla_{\partial_l}\partial_j \\ &= \nabla_{\partial_k}(\nabla_{\partial_j}\partial_l - \nabla_{\partial_l}\partial_j) + \nabla_{\partial_j}(\nabla_{\partial_l}\partial_k - \nabla_{\partial_k}\partial_l) + \nabla_{\partial_l}(\nabla_{\partial_k}\partial_j - \nabla_{\partial_j}\partial_k) \\ &= 0, \end{aligned}$$

since $T = 0$.

Correction. Page 13, Line 5:
`Reads as:` identities for tensors, it suffices
`Should read as:` identities for tensor fields, it suffices

Correction. Page 15, Line -9:
`Reads as:` $-\langle Z, [Y,X]\rangle + \langle [Y,Z], X\rangle - \langle Y, [X,Z]\rangle\}$,
`Should read as:` $-\langle Z, [Y,X]\rangle - \langle [Y,Z], X\rangle - \langle Y, [X,Z]\rangle\}$,

Correction. Page 17, Line -10:
`Reads as:` a map $\phi : A \to N \in C^k, k \geq 1$, if there exists an open
`Should read as:` a map $\phi : A \to N$ is C^k on A, $k \geq 1$, if there exist an open

Addition. Page 23, Line -3: `Add the proof of Corollary 1.2:` It suffices to show that for every $\tau \in (0, \ell)$ there exists $\delta > 0$ such that $\omega|[\tau, \tau + \delta]$ is a geodesic. To prove the existence of δ, given $\omega(\tau)$, let ϵ be given by Theorem 1.6 for $p = \omega(t)$, and set $\delta = \min\{\epsilon/2, \ell - \tau\}$. Then there exists a unique geodesic $\gamma : [\tau, \tau + \delta] \to M$, $|\gamma'| = 1$ from $p = \omega(\tau) = \gamma(\tau)$ to $\gamma(\tau + \delta) = \omega(\tau + \delta)$ of length equal to the distance from $\omega(\tau)$ to $\omega(\tau + \delta)$, namely, of length equal to δ. This implies $\delta = \ell(\omega|[\tau, \tau + \delta]) \geq \ell(\omega|[\tau, \tau + \delta]) = \delta$. Therefore $\omega|[\tau, \tau + \delta] = \omega|[\tau, \tau + \delta]$. Since both paths are parametrized with respect to arc length based at p, they must be identical. qed

Correction. Page 28, Line 8:
`Reads as:` on U.
`Should read as:` on U. We let $\{\omega^1, \ldots, \omega^n\}$ denote the dual coframe field.

Correction. Page 29, Line 9:
`Reads as:` $\sum_k \{Y({\omega_j}^k(X))e_k + \sum_k {\omega_j}^k(X)\nabla_Y e_k\}$
`Should read as:` $\sum_k \{Y({\omega_j}^k(X))e_k + {\omega_j}^k(X)\nabla_Y e_k\}$

Correction. Page 29, Line 15:
`Reads as:` ${\omega_k}^l(X, Y)\big\}$.
`Should read as:` ${\omega_k}^l(X, Y)\big\}\, e_k$.

Correction. Page 30, Line 18:
`Reads as:` …intrinsic Riemann geometry of surfaces.
`Should read as:` …intrinsic Riemannian geometry of surfaces.

Sketch of Solution. Page 33, Exercise 1.3 Since $\operatorname{grad} f$ has constant length, one certainly has $\operatorname{grad} f \perp \nabla_{\operatorname{grad} f} \operatorname{grad} f$. Therefore, given $p \in M$, $\xi \in M_p$, $\xi \perp (\operatorname{grad} f)_{|p}$,, we wish to show $\xi \perp \nabla_{\operatorname{grad} f} \operatorname{grad} f$. Let $\omega(t)$ be the integral curve of $\operatorname{grad} f$ satisfying $\omega(0) = p$, and X the parallel vector field along ω satisfying $X(0) = \xi$. Then one easily shows that

$$\langle \nabla_{\operatorname{grad} f} \operatorname{grad} f, \xi \rangle = \langle \nabla_t \omega', X \rangle = 0.$$

Correction. Page 39, Line 12:
`Reads as:` $= -\rho^2$
`Should read as:` $= -1/\rho^2$.

Correction. Page 40, Line 19:
`Reads as:` X is π-related to X.
`Should read as:` X is π-related to X. We say that a vector field X *on* M *projects to* N if there exists a vector field X on N which is π-related to X.

Correction. Page 40, Line 22:

`Reads as:` if X, Y, Z are horizontal vector fields, and T
`Should read as:` if X, Y, Z are horizontal vector fields all of which π-project, and T

Correction. Page 41, Line 2:
`Reads as:`

$$x(p) = 0, \qquad \pi_k \circ x = y \circ \pi.$$

`Should read as:`

$$x(p) = 0, \qquad \pi_k \circ x = y \circ \pi.$$

Also, show that for any vector field X on N there exists a horizontal lift of X to M, that is, a vector field X on M π-related to X.

Hint. Page 41, Exercise 1.16(b) Use Exercise 1.14(d) and the properties of the Levi-Civita connection to show that ${\Gamma_{rs}}^{\alpha}$ is skew-symmetric with respect to r, s.

Correction. Page 46, Line -8:
`Reads as:` In particular, there exists
`Should read as:` In particular, π is a submersion, and there exists

Correction. Page 47, Line -5:
`Reads as:` submersions (by the way, the
`Should read as:` submersions (the

Chapter II: Riemannian curvature

Correction. Page 55, Line -3:
`Reads as:` $\{\overline{\omega}_1, \ldots, \overline{\omega}_m\}$
`Should read as:` $\{\overline{\omega}^1, \ldots, \overline{\omega}^m\}$

Correction. Page 62, Line 4:
`Reads as:` $p, q, \in \mathbb{S}^n$
`Should read as:` $p, q \in \mathbb{S}^n$

Correction. Page 62, Line 17:
`Reads as:` $A_* \gamma'(0) = \gamma'(0)$ which
`Should read as:` $A_* \gamma'(0) = \gamma'(0)$, which

Correction. Page 65, Line -9:
`Reads as:` $\mathcal{K}(\alpha, \beta) = -1$
`Should read as:` $\mathcal{K}(\partial_\alpha, \partial_\beta) = -1$

Correction. Page 65, Line -5:
`Reads as:` $\mathcal{K}(\alpha, n) = -1$
`Should read as:` $\mathcal{K}(\partial_\alpha, \partial_n) = -1$

Correction. Page 69, Line 13:
`Reads as:` $< \nabla_t \partial_\epsilon \nu, \nabla_t \nu / |\partial_t \nu| >$
`Should read as:` $< \nabla_t \partial_\epsilon \nu, \partial_t \nu / |\partial_t \nu| >$

Correction. Page 71, Line 1:
`Reads as:` $\mathcal{L} =$
`Should read as:` $\mathcal{L}X =$

Correction. Page 77, Line 2:
`Reads as:` $+0(\lambda^2)$
`Should read as:` $+O(\lambda^2)$

Correction. Page 77, Line 3:
`Reads as:` $+0(\lambda^2)$
`Should read as:` $+O(\lambda^2)$

Correction. Page 77, Line 4:
`Reads as:` $+0(\lambda^2)$
`Should read as:` $+O(\lambda^2)$

Correction. Page 77, Line 5:

Reads as: $+0(\lambda^2)$
Should read as: $+O(\lambda^2)$

Correction. Page 81, Line 3–4:
Reads as: $(0, t_0]$, we have E is parallel along $\gamma|[0, t_0]$.
Should read as: $(0, t_0]$, and E is a unit vector field, we have E is parallel along $\gamma|[0, t_0]$.

Addition. Page 83, Line 3: We add here some indication of the calculation. The idea is that Taylor's expansion, in a neighborhood of $t = 0$, for a vector field $Y(t)$ along the geodesic $\gamma_\xi(t)$ is given by

$$Y(t) = \tau_t\{Y(0) + t\nabla_t Y(0) + (t^2/2){\nabla_t}^2 Y(0) + (t^3/6){\nabla_t}^3(0) + O(t^4)\},$$

where τ_t denotes parallel translation along γ_ξ from p to $\gamma_t(\xi)$. Now one uses the hypotheses of the theorem to calculate the derivatives of $Y(t)$ at $t = 0$. In the inner product, one uses the fact that the parallel translation is an isometry.

Correction. Page 83, Line -6ff:
Reads as: (2.43) and (2.62)
Should read as: (2.62) and Proposition 2.7

Addition. Page 84, Line 10: We add here some indication of the proof of Proposition 2.8. It suffices to show that if $A(t)$ is a differentiable map of the reals to the space of $n \times n$–matrices (with obvious differentiable structure — see Example 1.4(1)) satisfying $A(0) = I$, the identity matrix, then the Taylor expansion, about $t = 0$, of the function $t \mapsto \det A(t)$ is given by

$$\det A(t) = 1 + t \operatorname{tr} A'(0) + O(t^2).$$

This is a direct consequence of the definition of the determinant. One then has to adjust for $A(0)$ not necessarily equal to the identity matrix I.

Correction. Page 86, Line 19:
Reads as: any u_j, $j = 1, \ldots$
Should read as: any $u_j \in M_p$, $j = 1, \ldots$

Correction. Page 88, Line 13ff:
Should read as: **Exercise 2.3** (a) Let M be a submanifold of $\overline{M}$ as described in §2. We say that M is *totally geodesic in* $\overline{M}$, if for any geodesic γ in $\overline{M}$, for which there exists t_0 such that $\gamma(t_0) \in M$, there exists an $\epsilon > 0$ such that $\gamma|(t_0 - \epsilon, t_0 + \epsilon)$ is completely contained in M. Show that M is totally geodesic if and only if the second fundamental form B vanishes identically on M.

Correction. Page 88, Line -1:
Reads as: strictly negative
Should read as: nonpositive

Hint. Page 88, Exercise 2.5 Since the surface M is compact, it contained in some open 3–disk, $\mathbb{B}(o;R)$. Now keep $o \in \mathbb{R}^3$ fixed, and decrease R towards 0, and consider what happens when $R = r_o =: \inf\{r : \mathrm{M} \subset B(o;r)\}$.

Correction. Page 89, Line -8:
`Reads as:` minimal surfaces
`Should read as:` minimal submanifolds

Correction. Page 90, Line 2:
`Reads as:` connection in the normal bundle $\mathcal{D}$
`Should read as:` connection $\mathcal{D}$ in the normal bundle

Correction. Page 90, Line 6:
`Reads as:` of $\mathcal{D}$ we have
`Should read as:` of $\mathcal{D}$ (see §1.9) we have

Correction. Page 90, Line 8:
`Reads as:` $= \langle R(\xi,\eta)\sigma,\tau\rangle$
`Should read as:` $= \langle \overline{R}(\xi,\eta)\sigma,\tau\rangle$

Correction. Page 90, Line -3:
`Reads as:` $(\mathrm{Hess}\, f)_{|p} =$
`Should read as:` $(\mathrm{Hess}\, f)_{|(f^{-1}[\alpha])_p} =$

Correction. Page 93, Line -4:
`Reads as:` $\mathcal{D}\eta = \sum\limits_{k,\alpha}$
`Should read as:` $\mathcal{D}\eta = \sum\limits_{\alpha,\beta}$

Correction. Page 93, Line -2:
`Reads as:` $\otimes\omega^k \otimes e_\alpha$
`Should read as:` $\otimes\omega^j \otimes e_\alpha$

Correction. Page 94, Line 7:
`Reads as:` $S\mathcal{O}(n+1)/S\mathcal{O}(n)$
`Should read as:` $\mathcal{S}\mathcal{O}(n+1)/\mathcal{S}\mathcal{O}(n)$

Correction. Page 94, Line 15:
`Reads as:` curvature $-\rho^2$.
`Should read as:` curvature $-1/\rho^2$.

Correction. Page 96, Line 16:
`Reads as:` $\gamma|(0,\beta]$ has no conjugate points

`Should read as:` $\gamma|(0,\beta]$ is one-to-one and has no conjugate points

Correction. Page 96, Line -7:
`Reads as:` for r sufficiently small,
`Should read as:` for $r > 0$ sufficiently small,

Correction. Page 96, Line -6:
`Reads as:` for β sufficiently small,
`Should read as:` for $\beta > 0$ sufficiently small,

Correction. Page 97, Line 7:
`Reads as:` called *strongly convex* if
`Should read as:` called *convex* if

Correction. Page 97, Line 12:
`Reads as:` than “strongly convex” –
`Should read as:` than “convex” –

Correction. Page 97, Line 17:
`Reads as:` a strongly convex open
`Should read as:` a convex open

Chapter III: Riemannian volume

Correction. Page 101, Line 4:
`Reads as:` emanating from a point p, the cut point of p along the geodesic γ_ξ is the point
`Should read as:` at a point p, the cut point of p along the geodesic γ_ξ emanating from p is the point

Correction. Page 101, Line 13:
`Reads as:` curvature, then knowledge
`Should read as:` curvature -1, then knowledge

Correction. Page 101, Line 16:
`Reads as:` volume and geometry,
`Should read as:` volume and topology,

Correction. Page 109, Line -3: `The box  qed (concluding the proof of Theorem 3.4) should be at the end of the last line on the page, at the far right of line -1.`

Correction. Page 116, Line -4: `The proof of Proposition 3.2 should read as follows:` We first note that $r < R$ implies

$$\mathsf{D}_x(R) \subseteq \mathsf{D}_x(r).$$

Then

$$\begin{aligned}\frac{V(x;r+\epsilon)-V(x;r)}{\epsilon} &= \frac{1}{\epsilon}\int_r^{r+\epsilon} \mathfrak{A}(x;s)\,ds \\ &= \frac{1}{\epsilon}\int_r^{r+\epsilon} ds \int_{\mathsf{D}_x(s)} \sqrt{\mathbf{g}}(s;\xi)\,d\mu_x(\xi) \\ &\leq \frac{1}{\epsilon}\int_r^{r+\epsilon} ds \int_{\mathsf{D}_x(r)} \sqrt{\mathbf{g}}(s;\xi)\,d\mu_x(\xi) \\ &= \int_{\mathsf{D}_x(r)} d\mu_x(\xi) \frac{1}{\epsilon}\int_r^{r+\epsilon} \sqrt{\mathbf{g}}(s;\xi)\,ds.\end{aligned}$$

Let $\epsilon \downarrow 0$. Then Lebesgue's dominated convergence theorem implies the claim. qed

Correction. Page 118, Line 14:
`Reads as:` on all $t \in [(0,t_0]$.
`Should read as:` for all $t \in [(0,t_0]$.

Correction. Page 118, Line 16:
`Reads as:` $(\mathcal{A}^*\mathcal{A}(t;\xi)) \geq$

`Should read as:`
$(\mathcal{A}^*\mathcal{A})(t;\xi) \geq$

Correction. Page 119, Line 13:
`Reads as:` by Proposition 2.8
`Should read as:` by Proposition 2.8 and (3.27),

Correction. Page 120, Line 3:
`Reads as:` valid up to the cut point along each geodesic.
`Should read as:` valid up to the first conjugate point along each geodesic.

Definition. Given a geodesic $\gamma_\xi : [0,\beta) \to M$ in the Riemannian manifold M, $p = \gamma_\xi(0)$, $\xi = \gamma_\xi{}'(0)$, the *first conjugate point of* p *along* γ_ξ, $\gamma(t_o)$, is the point for which t_o is the infimum of all t' for which $\gamma_\xi(t')$ is conjugate to p along γ_ξ. We denote t_o by $\operatorname{conj}\xi$.

The function $(t,\xi) \mapsto \mathcal{A}(t;\xi)$ is continuous on $[0,\infty) \times \mathsf{S}M$, and $\mathcal{A}(0;\xi) = I$ for all ξ. The zeroes of $\mathcal{A}(;\xi)$, for each fixed ξ, characterize the conjugate points of $p = \pi(\xi)$ along γ_ξ. So for each ξ, $\operatorname{conj}\xi$ is indeed a minumum (not just an infimum) and $\operatorname{conj}\xi > 0$.

Correction. Page 120, Line 4:
`Reads as:` We assume
`Should read as:` Assume

Correction. Page 120, Line 8,10,12,13:
`Reads as:` $c(\xi)$
`Should read as:` $\operatorname{conj}\xi$

Correction. Page 120, Line 1:8 `Delete the last sentence of Remark 3.2.`

Correction. Page 122, Line 6: `until the end of the proof of the Theorem on page 123.`
Assume that $\Phi > 0$ on all of $(0,t)$, $t \in (0, \operatorname{conj}(\xi))$. Then the inequality (3.42) implies

$$\frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)\kappa} \geq 1, \tag{0.43}$$

which implies

$$\int_0^s \frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)\kappa}(\tau)\, d\tau \geq s \quad \forall \quad s \in (0,t]. \tag{0.44}$$

That is,

$$\mathbf{arcCt}_\kappa \frac{\phi(s)}{(n-1)} \geq s \quad \forall \quad s \in (0,t],$$

which implies

$$\phi \leq \psi \qquad \text{on } [(0,t],$$

which is (3.36). Of course, (3.37) follows easily.

If we have equality in (3.36) at some $t_0 \in (0, t]$, then the equality in (3.44) at $s = t_0$ implies we have equality in (3.41) and (3.42) on all of $(0, t_0]$. This, in turn, implies

$$\phi = \psi, \qquad \operatorname{tr} \mathcal{R} = (n-1)\kappa,$$

and $\mathcal{U}$ is a scalar multiple of the identity for each $s \in (0, t_0]$. Since $\mathcal{U}$ is a scalar multiple of the identity at each t, the Riccati equation (3.40) implies that is $\mathcal{R}$ is a scalar multiple of the identity for each s. This implies $\mathcal{R}(s) = \kappa I$ for all $s \in (0, t_0]$. Finally, since $\mathcal{U}$ is a scalar multiple of the identity and its trace is identically equal to $(n-1)\mathbf{S}_\kappa{}'(s)/\mathbf{S}_\kappa(s)$, we have

$$\mathcal{A}'\mathcal{A}^{-1}(s) = \frac{\mathbf{S}_\kappa{}'(s)}{\mathbf{S}_\kappa(s)} I$$

for all $s \in (0, t_0]$. But this then implies $\mathcal{A}(s) = \mathbf{S}_\kappa(s) I$ for all $s \in (0, t_0]$, which is (3.38).

Now let t be arbitrary in $(0, \operatorname{conj} \xi]$, and assume we do not have $\phi \leq \psi$ on all of $(0, t)$. Then there exists a maximal $t_1 \in (0, t)$ such that $\phi \leq \psi$ on $(0, t_1)$. In particular, $\phi = \psi$ at t_1. Then $\Phi(t_1) > 0$ and there exists $\epsilon_1 > 0$ such that $\Phi|[t_1, t_1 + \epsilon_1) > 0$, which implies (0.43) is valid from t_1 to any $s \in (t_1, t_1 + \epsilon_1)$, which implies $\phi \leq \psi$ on $(t_1, t_1 + \epsilon_1)$ — a contradiction to the maximality of t_1. So we have (3.36) on all of $(0, t]$.

To consider the case of equality, it suffices to consider the case where there exists $t_2 \in [0, t]$ such that $\phi < \psi$ on $(0, t_2)$ and $\phi(t_2) = \psi(t_2)$. But then $\Phi(t_2) = \Psi(t_2) > 0$, which implies there exists $\epsilon > 0$ such that (3.43) is valid on $(t_2 - \epsilon, t_2]$. For any $t \in (t_2 - \epsilon, t_2]$, integrate (3.43) from t to t_2. One obtains $\phi(t) \geq \psi(t)$ — a contradiction. qed

Correction. Page 128, Line 17:
`Reads as:` Then the coarea formula yields
`Should read as:` The Proposition 3.1 and the coarea formula yield

Correction. Page 129, Line 10:
`Reads as:` integral curves on
`Should read as:` integral curves of $\operatorname{grad} r$ on

Correction. Page 134, Line 16:
`Reads as:` $\mathsf{C}(\mathfrak{M})$
`Should read as:` $\nu\mathsf{C}(\mathfrak{M})$

Correction. Page 134, Line -5:
`Reads as:` pp.106ff.).
`Should read as:` pp.105ff.).

Correction. Page 139, Line 6:
`Reads as:` be a nowhere vanishing.

Should read as: vanish nowhere.

Correction. Page 141, Line 5:
Reads as: $= \sum_{j=1}^{n} (-1)^{j-1} d(\sqrt{g}\xi^j)\, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$
Should read as: $= \sum_{j=1}^{n} (-1)^{j-1} d(\sqrt{g}\xi^j) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$

Correction. Page 143, Line -2:
Reads as: $\mathsf{i}(\xi)\sigma = \sum_j (-1)^{j-1} \langle \xi, e_j \rangle \omega^1 \wedge \cdots \widehat{\omega^j} \wedge \cdots \wedge \omega^n$.
Should read as: $\mathsf{i}(\xi)\sigma = \sum_j (-1)^{j-1} \langle \xi, e_j \rangle \omega^1 \wedge \cdots \wedge \widehat{\omega^j} \wedge \cdots \wedge \omega^n$.

Correction. Page 147, Line -5:
Reads as: $\tau_g(p_j)$
Should read as: $g \cdot p_j$

Correction. Page 147, Line -4:
Reads as: $B(p; r)$
Should read as: $S(p; r)$

Correction. Page 149, Line -6:
Reads as: $\mathbf{n}^* d\mu_2 = \mathcal{K}\, dA$.
Should read as: $\mathbf{n}^* \sigma_{\mathbb{S}^2} = \mathcal{K}\, \sigma$. where σ is the area 2–form on M, and $\sigma_{\mathbb{S}^2}$ is the area 2–form on $\mathbb{S}^2$.

Correction. Page 149, Line -3:
Reads as: Prove that
Should read as: Recall (Exercise 2.5) that

Correction. Page 150, Line -9:
Reads as: $2\boldsymbol{\omega}_{\mathbf{n}} R$
Should read as: $\mathbf{c}_{\mathbf{n-1}}$

Correction. Page 152, Line -11:
Reads as: map I of $S(\mathfrak{M}; t)_{\gamma_\xi(t)}$.
Should read as: map I of $S(\mathfrak{M}; t)_{\gamma_\xi(t)}$, as $t \uparrow +\infty$.

Correction. Page 156, Line 6:
Reads as: for f, h in M.
Should read as: for f, h in $L^2(M)$.

Correction. Page 158, Line -7:
Reads as: For $k > 1$,
Should read as: Then

Correction. Page 162, Line -11:
`Reads as:` $\mathcal{E}$,
`Should read as:` $\mathcal{E}_o$

Correction. Page 163, Line 4:
`Reads as:` $\lambda_\kappa(R)$. Since
`Should read as:` $\lambda_\kappa(R)$, so either $F = 0$ or $G = 0$. Since

Correction. Page 163, Line 5:
`Reads as:` never vanishes, we have $G = 0$
`Should read as:` never vanishes, but $\int G\,dA = 0$ over every $\mathbb{S}_\kappa(o; r)$, we have $G = 0$

Correction. Page 170, Line 6:
`Reads as:` $\langle \mathfrak{x}, \mu \rangle$
`Should read as:` $\langle \mathfrak{x}, \nu \rangle$

Chapter IV: Riemannian coverings

Correction. Page 173, Line -9: `Replace the entire discussion starting at Proposition 4.1 through the conclusion of the proof of Theorem 4.3 (bottom p. 174) by the following:`

Proposition 4.1 *If* $\psi : \widetilde{M} \to M$ *is a Riemannian covering, then* M *is complete if and only if* $\widetilde{M}$ *is complete.*

Proof. One uses the fact that

$$\exp \psi_{*}\widetilde{\xi} = \psi(\exp \widetilde{\xi}) \tag{0.45}$$

for all $\widetilde{\xi} \in \mathcal{T}\widetilde{M}$ (see the generalities on isometries in §II.3) as follows:

If $\widetilde{M}$ is complete, let $\gamma : I \to M$ be a maximal geodesic with interval I containing the origin of $\mathbb{R}$. Pick $\widetilde{p}$, $\widetilde{\xi} \in \widetilde{M_{\widetilde{p}}}$ so that $\phi(\widetilde{p}) = \gamma(0)$, $\phi_{*|\widetilde{p}}\widetilde{\xi} = \gamma'(0)$. Then $\phi(\widetilde{\gamma}_{\widetilde{\xi}}(t)) = \gamma(t)$ for all $t \in I$. But $\phi(\widetilde{\gamma}_{\widetilde{\xi}}(t))$ is defined for all $t \in \mathbb{R}$. This implies (since I is maximal) that $I = \mathbb{R}$. So every geodesic in M is infinitely extendable in both directions. Therefore M is complete.

If M is given to be complete, and $\widetilde{\gamma}(t)$, $t \in \widetilde{I}$ is a maximal geodesic in $\widetilde{M}$, then $\gamma(t) := \phi(\widetilde{\gamma}(t))$ is a geodesic in M. But γ can be defined on all of $\mathbb{R}$. The unique lifting lemma will then imply that $\widetilde{\gamma}$ can be defined on all of $\mathbb{R}$. qed

Theorem 4.1 (S.B. Myers **[225]** *If* M *satisfies the hypotheses of the Bonnet–Myers Theorem* (Theorem II.12)*, that is, if* M *is complete and the Ricci curvature of* M *is bounded from below by a positive constant, then not only is* M *compact but any cover of* M*,* $\widetilde{M}$*, is also compact.*

Proof. Exercise for the reader. qed

Proposition (added) *Let* X*,* Y *be Riemannian manifolds,* $\phi : X \to Y$ *a local isometry. Then for any* $p \in X$*, there exists an* $\epsilon > 0$ *such that*

$$\phi|B(p;\epsilon) : B(p;\epsilon) \to B(\phi(p);\epsilon)$$

is an isometry.

Proof. Let ϵ_p and $\epsilon_{\phi(p)}$ satisfy

$$\begin{aligned} \exp|\mathsf{B}(p;\epsilon_p) : \mathsf{B}(p;\epsilon_p) &\to B(p;\epsilon_p) \\ \exp|\mathsf{B}(\phi(p);\epsilon_{\phi(p)}) : \mathsf{B}(\phi(p);\epsilon_{\phi(p)}) &\to B(\phi(p);\epsilon_{\phi(p)}) \end{aligned}$$

be diffeomorphisms, and pick $\epsilon = \min\{\epsilon_p, \epsilon_{\phi(p)}\}$. The (Euclidean) disks $\mathsf{B}(p;\epsilon)$ and $\mathsf{B}(\phi(p);\epsilon)$ are isometric under $\phi_{*|p}$, and

$$\phi|B(p;\epsilon) = \{\exp|\mathsf{B}(\phi(p);\epsilon)\}\circ\phi_{*|p}\circ\{\exp|\mathsf{B}(p;\epsilon)\}^{-1}$$

is, therefore, a diffeomorphism, which implies the proposition. qed

Theorem 4.2 *Let $\widetilde{M}$ be connected and complete, and let $\psi : \widetilde{M} \to M$ be a local isometry of $\widetilde{M}$ onto M. Then ψ is a covering.*

Proof. M is certainly connected since ψ is continuous.

Let $\gamma : [0, T_o] \to M$ be any geodesic segment in M, $\widetilde{p} \in \widetilde{M}$ such that $\psi(\widetilde{p}) = \gamma(0)$. Then γ has a unique lift in $\widetilde{M}$ starting at $\widetilde{p}$. Indeed, consider $\widetilde{\xi} \in \mathcal{T}\widetilde{M}$ such that $\psi_{*|\widetilde{p}}\widetilde{\xi} = \gamma'(0)$. Then (by Proposition (added)) there exists an $\epsilon > 0$ such that $\widetilde{\gamma}_{\widetilde{\xi}}(t)$ is defined for $t \in [0, \epsilon)$. Therefore, if we set

$$T = \sup\{\tau : \ \gamma|[0,\tau] \text{ has lift starting at } \widetilde{p}\},$$

then $T > 0$ and $\psi(\gamma_{\widetilde{\xi}}(t)) = \gamma(t)$ for all $t \in [0, T)$; also

$$\lim_{t \uparrow T} \gamma(t) = \psi(\widetilde{\gamma}_{\widetilde{\xi}}(T)).$$

If $T < T_o$, the lift can defined at T and beyond, which implies $T = T_o$.

Next, suppose we are given $p \in M$; we wish to construct a connected open neighborhood U of p such that ψ maps each component of $\psi^{-1}[U]$ homeomorphically onto U. To this end, fix $\epsilon > 0$ so that $\exp|\mathsf{B}(p;\epsilon)$ is a diffeomorphism of $\mathsf{B}(p;\epsilon)$ onto $B(p;\epsilon)$.

Now one shows

$$\psi^{-1}[B(p;\epsilon)] = \bigcup_{\widetilde{p} \in \psi^{-1}[p]} B(\widetilde{p};\epsilon).$$

Indeed, if

$$\widetilde{q} \in \bigcup_{\widetilde{p} \in \psi^{-1}[p]} B(\widetilde{p};\epsilon),$$

then there exists a path $\widetilde{\omega}$ joining some $\widetilde{p} \in \psi^{-1}[p]$ to $\widetilde{q}$ having length less than ϵ. This implies that $\omega := \psi(\widetilde{\omega})$ joins p to $\psi(\widetilde{q})$ and has length less than ϵ. So $\psi(\widetilde{q}) \in B(p;\epsilon)$, that is, $\widetilde{q} \in \psi^{-1}[B(p;\epsilon)]$.

On the other hand, if $\widetilde{q} \in \psi^{-1}[B(p;\epsilon)]$, then $q := \psi(\widetilde{q}) \in B(p;\epsilon)$, which implies there exists a geodesic $\gamma_{q,p} : [0, T] \to M$ from q to p, with length less than ϵ. Then $\gamma_{q,p}$ has a lift $\widetilde{\gamma} : [0, T] \to \widetilde{M}$ starting at $\widetilde{q}$; so $\psi(\widetilde{\gamma}(T)) = p$, and $d(\widetilde{q}, \widetilde{\gamma}(T)) < \epsilon$, which implies

$$\widetilde{q} \in \psi^{-1}[B(p;\epsilon)],$$

which implies the claim.

Finally, by the triangle inequality, given $\widetilde{p}_1$, $\widetilde{p}_2 \in \psi^{-1}[p]$, $\widetilde{p}_1 \neq \widetilde{p}_2$, $\epsilon < d(\widetilde{p}_1, \widetilde{p}_2)$ one has

$$B(\widetilde{p}_1;\epsilon/3) \cap B(\widetilde{p}_2;\epsilon/3) = \emptyset.$$

But the claims of the preceding paragraph are also valid for when ϵ is replaced by $\epsilon/3$. Thus the desired neighborhood U about p is $B(p;\epsilon/3)$. qed

Corollary 4.1 *The map $\mathcal{E}^i : \mathbb{R} \to \mathbb{S}^1$ given by*

$$\theta \mapsto (\cos\theta, \sin\theta) =: \mathcal{E}^{i\theta}$$

is a covering.

Theorem 4.3 (J. Hadamard [**162**], E. Cartan [**63**]) *If M is complete, and all of its sectional curvatures are nonpositive, then for any $p \in M$, $\exp_p : M_p \to M$ is a covering.*

Proof. Let g denote the Riemannian metric on M, and consider the Riemannian metric $(\exp_p)^* g$ on M_p. Then straight lines emanating from the origin of M_p are geodesics in the Riemannian metric $(\exp_p)^* g$.

By Corollary 1.5, $(\exp_p)^* g$ is a complete Riemannian metric on M_p. the theorem now follows from Corollary 2.2 and Theorem 4.2. qed

Correction. Page 202, Line 18:
`Reads as:` $\omega^j = \sum_k$
`Should read as:` $d\omega^j = \sum_k$

Correction. Page 218, Line 12–13:
`Reads as:` is equal to 2π times its Euler characteristic.
`Should read as:` is equal to its Euler characteristic.

Chapter V: The kinematic density

Correction. Page 229, Line -5:
`Reads as:` σ is invariant
`Should read as:` τ is invariant

Correction. Page 231, Line -7:
`Should read as:` **Theorem V.5** (Santalo's formula) *For all integrable* F *on* $\mathsf{S}\Omega$ *we have*

$$\int_{-\mathsf{U}\Omega} F\,d\mu = \int_{\mathsf{S}^+\partial\Omega} \langle \xi, \overline{\nu}_{\pi(\xi)}\rangle\,d\sigma(\xi) \int_0^{\ell(\xi)} F(\Phi_t\xi)\,dt.$$

Furthermore, if $\tau < +\infty$ *on all of* $\mathsf{S}\Omega$ *then we also have*

$$\int_{\mathsf{S}\Omega} F\,d\mu = \int_{\mathsf{S}^+\partial\Omega} \langle \xi, \overline{\nu}_{\pi(\xi)}\rangle\,d\sigma(\xi) \int_0^{\tau(\xi)} F(\Phi_t\xi)\,dt.$$

Correction. Page 242, Line 1:
`Reads as:` all of M
`Should read as:` all of $\mathsf{S}M$

Correction. Page 246, Line 10:
`Reads as:` $\{|\nabla_t E_j|^2 - \langle R({\gamma_\xi}', E_j){\gamma_\xi}', E_j\rangle\}\,dt$
`Should read as:` $\{|\nabla_t X_j|^2 - \langle R({\gamma_\xi}', X_j){\gamma_\xi}', X_j\rangle\}\,dt$

Chapter VI: Isoperimetric Inequalities

Correction. Page 283, Line -3:
`Reads as:` for all $\epsilon > 0$, with equality for some $\epsilon_o > 0$ if and only if X is a metric disk.
`Should read as:` for all $\epsilon > 0$. (Note: while the claim for equality is true, it is not proven in the text.

Correction. Page 284, Line -8:
`Reads as:` and to each
`Should read as:` and for each

Correction. Page 284, Line -3:
`Reads as:` the $(n-1)$–dimensional disk
`Should read as:` the closed $(n-1)$–dimensional disk

Correction. Page 285, Line 1:
`Reads as:` Note that the map $y \mapsto A(X^y) = A(D^y)$ is upper semi-continuous. In particular, $\sigma_\Gamma(X)$is compact, and certainly (by (3.51))
`Should read as:` One verifies directly that $\sigma_\Gamma(X)$is compact, and (by (3.51))

Correction. Page 285, Line 12:
`Reads as:` Then this collection
`Should read as:` Then, by the Blaschke selection theorem, (see I. Chavel, *Isooperimetric Inequalities: Differential Geometric and Analytic Perspectives*, Cambridge U. Press, New York, 2001, pp. 55ff) the collection

Correction. Page 286, Line 20:
`Reads as:` Z by
`Should read as:` Z in M^{y_1} by

Correction. Page 288, Line 3:
`Reads as:` $3\alpha/2$
`Should read as:` $5\alpha/4$