BOTTLENECK CONSTANTS OF FINITE GRAPHS

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In this note we consider an example to test three lower bounds presented in Diaconis–Stroock [**4**] (one by Diaconis and Stroock (DS), another by Sinclair (S) [**9**], and the third by Jerrum–Sinclair (JS) [**8**]) of the first positive eigenvalue of a finite graph. The lower bounds are quite sharp in many of the examples studied in [**4**], and seem to measure the bottlenecks or complexity of the graphs studied. Our example here is quite simple, and is motivated by differential geometric considerations — the Cheeger–Calabi dumbbell [**3**] and the lower bounds for the bottom of the spectrum of a Riemann surface of finite volume [**6, 7**] — from which we construct a family of examples in which we can make all the above estimates as poor as we like (the precise sense to be explicated below).

The example is quite straightforward. One considers a graph all of whose points have one edge connecting each point to the other. Distinguish two points of the graph, take another copy of the graph with the same distinguished two points, connect the graphs with one edge connecting each of the two distinguished points to its copy in the second graph. Now fiddle with *distinct* weights for these two edges. The details follow.

Before proceeding, however, we note that these examples show that the *upper* bound for λ_1 in terms of the discrete Cheeger constant is sharp.

The setup

Let X be a finite set, $|X|$ its cardinality; and let $\pi(x)$, $P(x,y)$ denote the probability distribution (playing the role of the volume element), and transition probability functions, respectively. So

$$Q(x,y) = \pi(x)P(x,y) = \pi(y)P(y,x)$$

is the density for the "energy integral"

$$E[\phi,\psi] = \frac{1}{2}\sum_{x,y} Q(x,y)\{\phi(x) - \psi(y)\}^2$$

associated with the Laplacian $\Delta = I - P$, relative to the $L^2(\pi)$ inner product. We are interested in the $L^2(\pi)$–orthogonal complement of the constants; so λ_1, the lowest positive eigenvalue of the Laplacian, is the infimum of the discrete Rayleigh quotient

$$\phi \mapsto E[\phi,\phi]/V(\phi),$$

where

$$V(\phi) = \sum_x \phi^2(x)\pi(x) - \left\{ \sum_x \phi(x)\pi(x) \right\}^2.$$

Since π is assumed to be a probability distribution, we have $V(\phi)$ is the variance of ϕ, $Var\,\phi$, given by

$$Var\,\phi = \frac{1}{2}\sum_{x,y} \{\phi(x) - \phi(y)\}^2 \pi(x)\pi(y).$$

In our setting, here, the primary data is given by $Q(x,y)$. To obtain $\pi(x)$ and $P(x,y)$, one easily has

$$\pi(x) = \sum_y Q(x,y), \qquad \Rightarrow \qquad P(x,y) = Q(x,y)/\sum_z Q(x,z).$$

A graph structure on X is determined by assigning an edge $[x \sim y]$ to any x, y which satisfy $Q(x,y) > 0$. The neighbors $\mathsf{N}(x)$ of x consist of all $y \in X$ such that $Q(x,y) > 0$, equivalently, those y for which there exists an edge joining x to y. We *assume* that the graph is connected, that is, to any x, y in X there exists a sequence $x = x_0, x_1, \ldots, x_l = y$ such that $x_j \in \mathsf{N}(x_{j-1})$ for all $j = 1, \ldots, l$.

The bottleneck constants

The Diaconis–Stroock constant, is defined as follows: One picks a collection of paths $\Gamma = \{\gamma_{x,y} : x, y \in X\}$, where each $\gamma_{x,y}$ joins x to y, $\gamma_{x,y} = \gamma_{y,x}$, where we allow $\gamma_{x,y}$ to intersect itself, but not repeat any edges. Once Γ is chosen we define the *Diaconis–Stroock* (DS) *constant,* $\kappa = \kappa_\Gamma$ by

$$\kappa = \max_e \sum_{\gamma_{x,y} \ni e} \left\{ \sum_{[z \sim w] \in \gamma_{x,y}} Q([z \sim w])^{-1} \right\} \pi(x)\pi(y),$$

where e varies over all the edges of X, $\gamma_{x,y}$ varies over all paths in Γ which pass through the edge e, and $[z \sim w]$ varies over the edges of $\gamma_{x,y}$. Then the theorem of Diaconis–Stroock [**4**] states that

$$\lambda_1 \geq 1/\kappa. \tag{0.1}$$

Similarly, a variant estimate in [**4**] from Sinclair [**9**] goes as follows: For our family Γ of paths, define the *Sinclair constant* (S) K by

$$K = \max_e Q(e)^{-1} \sum_{\gamma_{x,y} \ni e} |\gamma_{xy}| \pi(x)\pi(y),$$

where $|\gamma_{xy}|$ denotes the number of edges in the path γ_{xy}. Again, one has the estimate

$$\lambda_1 \geq 1/K. \tag{0.2}$$

The third constant, for a family of paths Γ, is the *Jerrum–Sinclair constant,* $\mathcal{K}$, is defined by

$$\mathcal{K} = \max_e Q(e)^{-1} \sum_{\gamma_{x,y} \ni e} \pi(x)\pi(y).$$

Then

$$\lambda_1 \geq 1/8\mathcal{K}^2. \tag{0.3}$$

All the above are to be compared to Cheeger's inequality. Define a *separator* σ to be a collection of edges whose removal disconnects the graph into two subsets S_1, S_2, and define the *Cheeger constant* h by

$$h = \inf_{\sigma} \sum_{[x\sim y]\in\sigma} Q(x,y)/\min\{\pi(S_1), \pi(S_2)\}.$$

Then the discrete Cheeger inequality states that

$$\frac{h^2}{2} \leq \lambda_1 \leq 2h. \tag{0.4}$$

(See the original argument for the lower bound in Dodziuk [**5**], and the upper bound in Alon–Milman [**1**]. For Riemannian manifolds the corresponding lower bound goes back to Cheeger [**3**], and the corresponding upper bound to Buser [**2**].) Of the first three bottleneck constants only the JS constant is directly comparable to the Cheeger constant:

$$h \geq 1/2\mathcal{K}$$

— in fact, that is how one derives (0.3) (see Proposition 7 of [**4**]).
Another lower lound for λ_1, essentially that of Dodziuk–Pignataro–Randol–Sullivan [**6**], goes as follows: Given a function $\phi(x)$ such that

$$\sum_x \phi(x)\pi(x) = 0, \quad \sum_x \phi^2(x)\pi(x) = 1, \quad \max\,\phi \geq |\min\,\phi|.$$

Then if $a_1 < \ldots < a_L$ denote the values of ϕ, we have

$$1 \leq \max\,\phi \leq \max\,\phi - \min\,\phi = \sum_{j=1}^{L} a_j - a_{j-1},$$

which implies there exists $\ell \in \{1, \ldots, L\}$ such that

$$a_\ell - a_{\ell-1} \geq \frac{1}{L} \geq \frac{1}{|X|}.$$

So

$$\begin{aligned}
\lambda_1 = E[\phi,\phi] &= \frac{1}{2}\sum_{x,y} Q(x,y)\{\phi(x)-\phi(y)\}^2 \\
&\geq \sum_{\phi(x)\geq a_\ell,\ \phi(y)\leq a_{\ell-1}} Q(x,y)\{\phi(x)-\phi(y)\}^2 \\
&\geq \frac{1}{|X|^2} \sum_{\phi(x)\geq a_\ell,\ \phi(y)\leq a_{\ell-1}} Q(x,y).
\end{aligned}$$

If $Q(x, y) = \alpha$ for all neighbors x, y then, for the valence function $m(x)$ and associated volume $V = \sum_x m(x)$, we have $\pi(x) = \alpha m(x)$, which implies

$$\alpha = \frac{1}{V}, \quad \pi(x) = \frac{m(x)}{V}, \quad P(x, y) = \frac{1}{m(x)}.$$

Then

$$\lambda_1 \geq \frac{1}{|X|^2} \inf_\sigma \frac{|\sigma|}{V}, \tag{0.5}$$

where the σ varies over the separators of X. If, in addition, X is k–regular (as in some of the examples in [**4**]), then $V = k|X|$, and the lower bound given by the Diaconis–Stroock constant is sharper than that given by (0.5). Note that if k is close to $|X|$, then this estimate improves Cheeeger's inequality (0.5).

The example

Let $(X; x_1, x_2)$, be a 2–pointed finite graph ($x_1 \neq x_2$ distinct), $|X| = n >> 2$, $m(x) = n-1$. So every vertex is connected to every other, and $V = n(n-1)$.

Next, let $(Y; y_1, y_2)$, $(Z; z_1, z_2)$ be two copies of $(X; x_1, x_2)$, and W the graph consisting of the union of vertices and edges of Y and Z, with one edge from each y_j to z_j, $j = 1, 2$. For every edge e in Y or Z we define $\ell(e) = 1$; and for the edge $e_j = [y_j \sim z_j]$, $j = 1, 2$, joining y_j to z_j define $\ell(e_j) = \epsilon_j << 1$, where $\epsilon_1, \epsilon_2 > 0$. Also define

$$\epsilon = \epsilon_1 + \epsilon_2.$$

Define for the edge $[\xi \sim \eta]$ in W,

$$Q(\xi, \eta) = \frac{\ell([\xi \sim \eta])}{\sum_{[w_1 \sim w_2]} \ell([w_1 \sim w_2])} = \frac{\ell([\xi \sim \eta])}{2n(n-1) + \epsilon}.$$

Then

$$\pi(w) = \frac{n-1}{2n(n-1) + \epsilon} \qquad w \neq y_j, z_j \ j = 1, 2,$$

and

$$\pi(w) = \frac{n-1+\epsilon_j}{2n(n-1) + \epsilon} \qquad w = y_j, z_j \ j = 1, 2.$$

Start with the test function $\phi : W \to \{-1, 1\}$ defined by $\phi(w) = 1$ on Y, and $\phi(w) = -1$ on Z. Then

$$\sum_w \phi(w)\pi(w) = 0, \qquad \sum_w \phi^2(w)\pi(w) = 1,$$

and

$$E[\phi, \phi] = \frac{4\epsilon}{2n(n-1) + \epsilon}.$$

So

$$\lambda_1 \leq \frac{4\epsilon}{2n(n-1) + \epsilon}.$$

To obtain a lower bound we have to do a little more. Namely, let ϕ denote an $L^2(\pi)$–normalized eigenfunction of λ_1. Then, since the exchange of Y and Z, $\iota_{Y,Z} : W \to W$, preserves the Laplacian we have the decomposition $\phi = \phi^- + \phi^+$, with

$$\iota_{Y,Z}\phi^+ = \phi^+, \qquad \iota_{Y,Z}\phi^- = -\phi^-.$$

We let ψ denote either ϕ^+ or ϕ^-. Assume ψ is not identically 0. Then we may normalize ψ to have L^2–norm equal to 1, which implies

$$\sum_Y \psi^2(y)\pi(y) = \frac{1}{2},$$

which implies

$$\sum_Y \psi^2(y)\{n-1+\epsilon\} \geq n(n-1) + \epsilon/2.$$

So there exists $\xi \in Y$ such that

$$\psi^2(\xi) \geq \frac{n(n-1)+\epsilon/2}{n(n-1+\epsilon)}.$$

Now if $\psi = \phi^+$, then

$$\sum_Y \psi(y)\pi(y) = 0,$$

which implies there exists $\eta \in Y$ such that $\psi(\eta) < 0$. This implies

$$\lambda_1 \geq \frac{1}{2n(n-1)+\epsilon}\{\psi(\xi) - \psi(\eta)\}^2\pi(\xi)\pi(\eta) \geq \text{const.} > 0,$$

the constant independent of ϵ, which contradicts the upper bound of λ_1 for sufficiently small $\epsilon > 0$. Therefore, for sufficiently small $\epsilon > 0$ we have $\phi^+ = 0$, and $\phi = \phi^-$.
We may assume ξ is the maximum point of ϕ; suppose $\xi \neq y_j, j = 1, 2$. Then

$$\frac{1}{n-1}\sum_{y\in \mathsf{N}(\xi)} \phi(y) = (1-\lambda_1)\phi(\xi) \geq \left(1 - \frac{4\epsilon}{2n(n-1)+\epsilon}\right)\phi(\xi),$$

which implies, for each $j = 1, 2$,

$$\left(1 - \frac{4\epsilon}{2n(n-1)+\epsilon}\right)\phi(\xi) \leq \frac{n-2}{n-1}\phi(\xi) + \frac{\phi(y_j)}{n-1},$$

which, in turn, implies

$$\phi(y_j) \geq \left\{1 - \frac{4(n-1)\epsilon}{2n(n-1)+\epsilon}\right\}\phi(\xi) \geq \left\{1 - \frac{4(n-1)\epsilon}{2n(n-1)+\epsilon}\right\}\sqrt{\frac{n(n-1)+\epsilon/2}{n(n-1+\epsilon)}}.$$

The corresponding estimate holds for each z_j, $j = 1, 2$.

Therefore

$$\begin{aligned}
\lambda_1 &= E[\phi,\phi] \\
&\geq \frac{1}{2n(n-1)+\epsilon}\sum_j \epsilon_j\{\phi(y_j)-\phi(z_j)\}^2 \\
&\geq \frac{4\epsilon}{2n(n-1)+\epsilon}\left\{1-\frac{4(n-1)\epsilon}{2n(n-1)+\epsilon}\right\}^2 \frac{n(n-1)+\epsilon/2}{n(n-1+\epsilon)} \\
&\geq \frac{2\epsilon}{n(n-1+\epsilon)}\left\{1-\frac{4(n-1)\epsilon}{2n(n-1)+\epsilon}\right\}^2.
\end{aligned}$$

Of course, if $\xi = y_j$ for some $j = 1, 2$, then the estimate is certainly true. Therefore,

$$\frac{2\epsilon}{n(n-1+\epsilon)}\left\{1-\frac{4(n-1)\epsilon}{2n(n-1)+\epsilon}\right\}^2 \leq \lambda_1 \leq \frac{4\epsilon}{2n(n-1)+\epsilon}. \tag{0.6}$$

So our estimate (0.6) for λ_1 is reasonably sharp.

Now consider the collection of paths Γ in W as follows: For ξ,η both in Y or both in Z, the path will consist of the edge from ξ to η. For each $\xi \in Y, \eta \in Z$, we pick a shortest path from ξ to η. The choice of shortest path between any $\xi \in Y$ and $\eta \in Z$ is not unique for all such ξ,η, so the idea is to pick these paths in such a way that the DS constant is as small as possible.

To every $j = 1, 2$, associate a subset $W_j \subseteq Y \times Z$, $|W_j| = n_j$,

$$n_1 + n_2 = n^2,$$

such that $W_1 \cap W_2 = \emptyset$. So W_1, W_2 is a partition of $Y \times Z$. We require that $\{y_j\} \times Z \subset W_j$, for $j = 1, 2$. In particular, $n_j \geq n$. For $(y,z) \in W_j$, where $y \neq y_j$ and $z \neq z_j$, we choose the minimal path $\gamma_{y,z}$ to be $y \to y_j \to z_j \to z$. If $y = y_j$ (resp., $z = z_j$), then leave out $y \to y_j$ (resp., $z \to z_j$).

For every u, v, $\gamma_{u,v} \in \Gamma$, we define

$$\mathcal{L}(\gamma_{u,v}) = \left\{\sum_{[w_1\sim w_2]\in\gamma_{u,v}} Q([w_1 \sim w_2])^{-1}\right\}\pi(u)\pi(v).$$

(a) We have, for $u, v \in Y$ or $u, v \in Z$,

$$\mathcal{L}(\gamma_{u,v}) \leq \frac{(n-1+\epsilon)^2}{2n(n-1)+\epsilon}.$$

(b) We have, for $(y,z) \in W_j$,

$$({\epsilon_j}^{-1})\frac{(n-1)^2}{2n(n-1)+\epsilon} \leq \mathcal{L}(\gamma_{y,z}) \leq (2+{\epsilon_j}^{-1})\frac{(n-1+\epsilon)^2}{2n(n-1)+\epsilon}$$

To estimate the DS constant we consider the possibilities:

If (i) ξ, η are both in Y or both in Z, $e = [\xi \sim \eta]$, and neither ξ nor η are from $\{y_j, z_j\}$, for any j, then

$$\sum_{\gamma_{u,v} \ni e} \mathcal{L}(\gamma_{u,v}) \leq \frac{(n-1+\epsilon)^2}{2n(n-1)+\epsilon}.$$

If (ii) $\xi \in Y$, $\xi \neq y_j$, then for $e = [\xi \sim y_j]$, we have

$$\sum_{\gamma_{u,v} \ni e} \mathcal{L}(\gamma_{u,v}) \leq \frac{(n-1+\epsilon)^2}{2n(n-1)+\epsilon} + n(2+{\epsilon_j}^{-1})\frac{(n-1+\epsilon)^2}{2n(n-1)+\epsilon}$$

We have a similar estimate if (ii) $\eta \in Z$, $\eta \neq z_j$, and $e = [z_j \sim \eta]$. For $e = [z_1 \sim z_2]$ we have

$$\sum_{\gamma_{u,v} \ni e} \mathcal{L}(\gamma_{u,v}) \leq \frac{(n-1+\epsilon)^2}{2n(n-1)+\epsilon} + n(4+{\epsilon_1}^{-1}+{\epsilon_2}^{-1})\frac{(n-1+\epsilon)^2}{2n(n-1)+\epsilon}$$

The last possibilty is (iii) $e = [y_j \sim z_j]$. Then

$$\sum_{\gamma_{u,v} \ni e} \mathcal{L}(\gamma_{u,v}) \geq {\epsilon_j}^{-1}\frac{(n-1)^2}{2n(n-1)+\epsilon} + n_j({\epsilon_j}^{-1})\frac{(n-1)^2}{2n(n-1)+\epsilon}$$

So, if n is large, $n_j \geq n^2/4$ for $j = 1, 2$, and ϵ is small, then we have κ realized by $e = [y_j \sim z_j]$ for which $n_j{\epsilon_j}^{-1}/2$ is the largest. If we have n_j approximately equal to $n^2/2$ for both $j = 1, 2$, then we are looking for the smallest value of ϵ_j. This implies

$$\kappa \sim \frac{n^2}{4\min_j\ \epsilon_j},$$

which implies

$$\lambda_1 \geq \frac{4\min_j\ \epsilon_j}{n^2};$$

while

$$\lambda_1 \sim \frac{2\epsilon}{n^2} = \frac{2}{n^2}(\epsilon_1+\epsilon_2).$$

So we can make the Diaconis–Stroock estimate as poor as we like. A similar estimate is valid for the Sinclair constant. The Jerrum–Sinclair estimate is of the order of the lower bound of the Cheeger inequality (0.4), but our example is sharp for the upper bound of (0.4). Note that if $\epsilon_1 = \epsilon_2$ then both Diaconis–Stroock and Sinclair give sharp estimates.

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